MULTIPLE INTEGRALS

I. Double Integrals:

(1) Evaluate each of the following double integrals, and sketch the region A over which the integration extends.

(a) $\int_0^{\pi}\int_0^{x} x \sin y \, dy\, dx$ (b) $\int_0^{\pi}\int_0^{\sin x} y \, dy\, dx$

(c) $\int_1^{2}\int_y^{y^2} dx\, dy$ (d) $\int_0^{2}\int_1^{e^x} dy\, dx$

(e) $\int_0^{1}\int_{\sqrt{y}}^{1} dx\, dy$ (f) $\int_{-2}^{1}\int_{x^2+4x}^{3x+2} dy\, dx$

(2) Evaluate $\iint_R dA$ where R is the region between $y = 2x$ and $y = x^2$ lying to the left of $x = 1$.

(3) Find the area of the region bounded by the parabola $x = y-y^2$ and the line $x+y = 0$.

(4) Using polar coordinates and double integration find (a) the total area enclosed by the lemniscate $r^2 = 2a^2\cos 2\theta$ and (b) the area that lies inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

(5) Find the center of gravity (letting $\delta(x,y) = 1$) of the area bounded by the coordinate axes and the line $x+y = a$.

(6) Find the center of gravity of the area bounded by the curve $y^2+x = 0$ and the line $y = x+2$ ($\delta(x,y) = 1$).

(7) Find the moment of inertia ($\delta = 1$) about the x-axis of the area bounded by the curve $y = e^x$ and the lines $x = 0$, $x = 1$.

(8) Find the polar moment of inertia about the z-axis for the area bounded by the x-axis, the curve $y = e^x$ and the lines $x = 0, x = 1$ ($\delta = 1$).

(9) Using polar coordinates, find the polar moment of inertia I_0 with respect to an axis through 0 perpendicular to the xy-plane for the area lying inside the cardioid $r = a(1+\cos\theta)$ and outside the circle $r = a$.

(10) Using double integration, find the following volumes: (a) in the 1st octant between the planes $z = 0$ and $z = x+y+2$ and inside the cylinder $x^2+y^2 = 16$. (b) bounded by the cylinder $x^2+y^2 = 4$ and the planes $y+z = 4$ and $z = 0$. (c) bounded above by the paraboloid $x^2+4y^2 = z$, below by the plane $z = 0$, and laterally by the cylinders $y^2 = x$ and $x^2 = y$. (d) of the wedge cut from the cylinder $4x^2+y^2 = a^2$ by the planes $z = 0$ and $z = my$.

(10e) Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4-x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x+4$.

(11) Using cylindrical coordinates, find the volume bounded by the paraboloid $x^2+y^2 = 4z$, the cylinders $x^2+y^2 = 8y$ and the plane $z = 0$.

II. Triple Integrals:

(1) By triple integration find the following volumes:

(a) of the tetrahedron bounded by the plane $x/a + y/b + z/c = 1$ and the coordinates planes.

(b) between the cylinder $z = y^2$ and the xy-plane that is bounded by the four vertical planes $x = 0$, $x = 1$, $y = -1$, $y = 1$.

(c) In the 1st octant bounded by the cylinder $x = 4-y^2$ and the planes $z = y$, $x = 0$, $z = 0$.

(d) Enclosed by the cylinder $y^2+4z^2 = 16$ and the planes $x = 0$, $x+y = 4$.

(e) Inside $x^2+y^2 = 9$, above $z = 0$ and below $x+z = 4$.

(2) Using cylindrical coordinates, find the volume:

(a) bounded above by the paraboloid $z = 5-x^2-y^2$ and below by the paraboloid $z = 4x^2+4y^2$.

(b) that is bounded above by the paraboloid $z = 9-x^2-y^2$, below by the xy-plane, and that lies outside the cylinder $x^2+y^2 = 1$.

(c) bounded below by the paraboloid $z = x^2+y^2$ and above by the plane $z = 2y$

(d) bounded above by the sphere $x^2+y^2+z^2 = 2a^2$ and below by the paraboloid $az = x^2+y^2$.

(3) Using spherical coordinates find the volume

(a) of the solid which lies above the cone $z^2 = x^2+y^2$ and inside the sphere $x^2+y^2+z^2 = 4z$.

(b) cut from the sphere $\rho = 2$ by the plane $z = \sqrt{2}$.

(c) enclosed by the surface $\rho = a\,(1-\cos\phi)$.

III. Applications of Triple Integration:

(1) Find the volume and centroid of the solid bounded by the graphs of $z = x^2+y^2$, $x^2+y^2 = 4$, and $z = 0$.

(2) Find the moment of inertia of a homogeneous rt. circular cylinder of altitude h and radius of base a with respect to each of the following:

(a) the axis of the cylinder.

(b) the diameter of the base.

(3) Find the mass and center of mass of a solid hemisphere of radius a if the density at a point P is directly proportional to the distance from the center of the base to P.

(4) Find the moment of inertia with respect to the axis of the hemisphere in the above problem.

(5) Find the moment of inertia about the x-axis for the volume cut from the sphere $x^2+y^2+z^2 = 4a^2$ by the cylinder $x^2+y^2 = a^2$.

(6) Use cylindrical coordinates to find the moment of inertia of a sphere of radius a and mass M about a diameter.

(7) Find the moment of inertia of a rt. circular cone of base radius a, altitude h, and mass M about an axis through the vertex and parallel to the base.

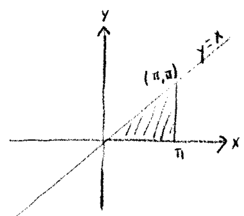
(8) Find the center of gravity of the volume (which resembles a filled ice-cream cone) that is bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/6$.

(9) Find the radius of gyration with respect to a diameter of a spherical shell of mass M bounded by the spheres $\rho = a$ and $\rho = 2a$ if the density is $\delta = \rho^2$.

I Double Integrals:

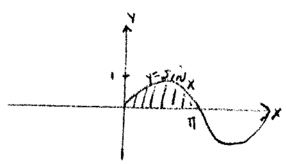
1. a) $\int_0^{\pi}\int_0^{x} x \sin y \, dy\, dx = \int_0^{\pi} \left[-x\cos y\right]_0^{x} dx$

$= \int_0^{\pi} (x - x\cos x)\, dx = \left[\frac{x^2}{2} - (x\sin x + \cos x)\right]_0^{\pi} = \frac{4+\pi^2}{2}$

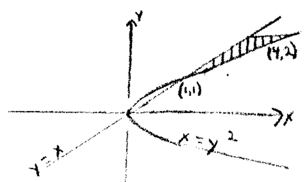


b) $\int_0^{\pi}\int_0^{\sin x} y\, dy\, dx = \int_0^{\pi} \sin^2 x/2 \, dx$

$= \frac{1}{2}\left[\frac{x - \sin x}{2}\right]_0^{\pi} = \pi/4$

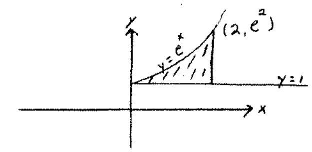


c) $\int_1^2 \int_y^{y^2} dx\, dy = \int_1^2 (y^2 - y)\, dy = 5/6$



d) $\int_0^2 \int_1^{e^x} dy\, dx = \int_0^2 y \Big|_1^{e^x} dx$

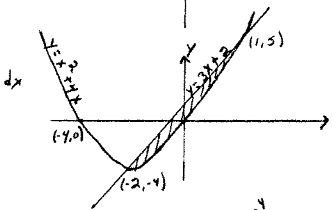
$= e^x - x \Big|_0^2 = e^2 - 3$



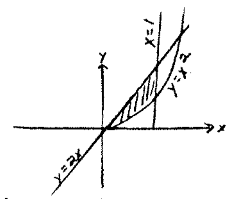
e) $\int_0^1 \int_{\sqrt{y}}^1 dx\, dy = \left(y - \frac{2}{3}y^{3/2}\right)\Big|_0^1 = 1/3$

f) $\int_{-2}^1 \int_{x^2+4x}^{3x+2} dy\, dx = \int_{-2}^1 \left[(3x+2) - (x^2+4x)\right] dx$

$= 9/2$



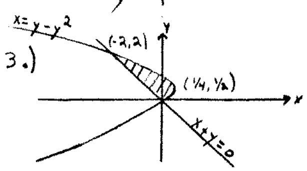
(2.)



$A = \int_0^1 \int_{x^2}^{2x} dy\, dx$

$= \int_0^1 (2x - x^2)\, dx = 2/3$

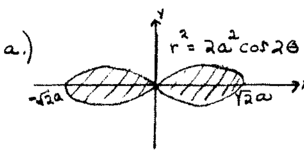
(3.)



$A = \int_0^2 \int_{-y}^{y-y^2} dx\, dy$

$= \int_0^2 (y-y^2) - (-y)\, dy = 4/3$

(4a.)

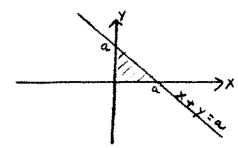


$A = 2\int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{2\cos 2\theta}} r\, dr\, d\theta$

$= 2\left(\frac{1}{2}\right) \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta$

$= 2a^2$

(5.)



$M = \iint \delta(x,y)\, dx\, dy$

$= \int_0^a \int_0^{a-x} (1)\, dy\, dx$

$= \frac{a^2}{2}$

$M_y = \iint x\, \delta(x,y)\, dx\, dy$

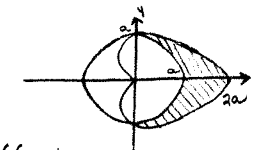
$= \int_0^a \int_0^{a-x} x\,(1)\, dy\, dx = \frac{a^3}{6}$

$M_x = \iint y\, \delta(x,y)\, dy\, dx$

$= \int_0^a \int_0^{a-x} y\, dy\, dx = a^3/6$

$\bar{x} = \frac{M_y}{M} = \frac{a^3/6}{a^2/2} = \frac{a}{3} \quad \bar{y} = \frac{M_x}{M} = \frac{a}{3}$

(4b)



$A = \iint r\, dr\, d\theta$

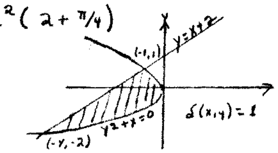
$= \int_{-\pi/2}^{\pi/2} \int_a^{a(1+\cos\theta)} r\, dr\, d\theta$

$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[a^2(1 + 2\cos\theta + \cos^2\theta) - a^2\right] d\theta$

$\boxed{\int \cos^2\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4}}$

$= a^2(2 + \pi/4)$

(6.)



$m = \int_{-2}^1 \int_{y-2}^{-y^2} dx\, dy = 9/2$

$M_y = \int_{-2}^1 \int_{y-2}^{-y^2} x\, dx\, dy = -\frac{36}{5}$

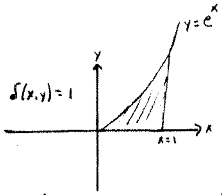
$M_x = \int_{-2}^1 \int_{y-2}^{-y^2} y\, dx\, dy = -\frac{27}{12}$

$\bar{x} = \frac{M_y}{m} = \frac{-36/5}{9/2} = -8/5$

$\bar{y} = \frac{M_x}{m} = \frac{-27/12}{9/2} = -1/2$

$(\bar{x}, \bar{y}) = (-8/5, -1/2)$

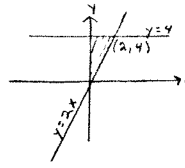
(7.)



$$I_x = \iint y^2\delta(x,y)\,dx\,dy = \int_0^1\int_0^{e^x} y^2\,dy\,dx$$

$$= \int_0^1 \frac{y^3}{3}\Big|_0^{e^x} dx = \frac19(e^3-1)$$

(8.)

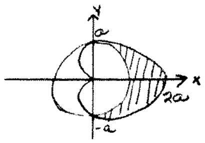


Axis thru O ⊥ xy plane

$$I_0 = \iint r^2\delta(x,y)\,dA = \int_0^2\int_{2x}^4 (x^2+y^2)\,dy\,dx$$

$$= \int_0^2 x^2y + \frac{y^3}{3}\Big]_{2x}^4 dx = 34\tfrac23$$

(9.)



$$I_0 = \iint r^2\,dA = \int_{-\pi/2}^{\pi/2}\int_a^{a(1+\cos\theta)} r^2\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\frac{r^4}{4}\Big|_a^{a(1+\cos\theta)}d\theta$$

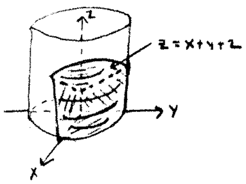
$$= \frac{a^4}{4}\int_{-\pi/2}^{\pi/2}\left[(1+\cos\theta)^4-1\right]d\theta = \frac{a^4}{4}\int_{-\pi/4}^{\pi/4}(\cos^4\theta+4\cos^3\theta+6\cos^2\theta+4\cos\theta+1-1)\,d\theta$$

$$= \frac{a^4}{4}\left\{\left(\frac{\cos^3\theta\sin\theta}{4}+\frac34\int\cos^2\theta\right)+4\left(\frac{\cos^2\theta\sin\theta}{3}+\frac23\int\cos\theta\right)+6\left(\frac{\cos\theta\sin\theta}{2}+\frac12\theta\right)+4\sin\theta\right\}_{-\pi/2}^{\pi/2}$$

$$\boxed{\int\cos^n\theta\,d\theta = \frac{\cos^{n-1}\theta\sin\theta}{n}+\frac{n-1}{n}\int\cos^{n-2}\theta\,d\theta}$$

$$= \frac{a^4}{4}\left\{\frac{\cos^3\theta\sin\theta}{4}+\frac34\left(\frac{\cos\theta\sin\theta}{2}+\frac12\theta\right)+\frac43\cos^2\theta\sin\theta+\frac83\sin\theta+3\cos\theta\sin\theta+3\theta+4\sin\theta\right\}_{-\pi/2}^{\pi/2} = \frac{a^4}{96}(81\pi+320)$$

(10a.)

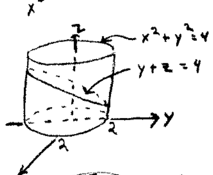


$$V = \iint f(x,y)\,dx\,dy = \int_0^4\int_0^{\sqrt{16-x^2}}(x+y+2)\,dy\,dx$$

$$= \int_0^4\left(x\sqrt{16-x^2}+\frac{16-x^2}{2}+2\sqrt{16-x^2}\right)dx$$

$$= \frac{128}{3}+8\pi \qquad \boxed{\int\sqrt{a^2-x^2} = \frac12 x\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C}$$

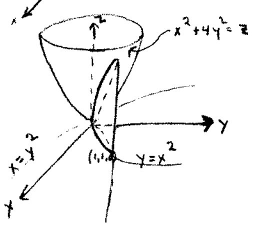
(10b.)



$$V/2 = \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\,dx = \int_0^2 4\left(2\sqrt{4-x^2}\right)-\left(\frac{4-x^2}{2}-\frac{4-x^2}{2}\right)dx$$

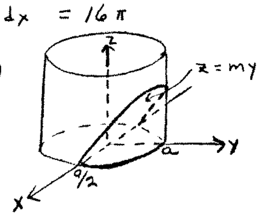
$$V = 16\int_0^2\sqrt{4-x^2}\,dx = 16\pi$$

(10c)



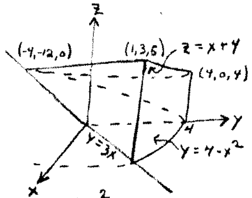
$$V = \int_0^1\int_{x^2}^{\sqrt x}(x^2+4y^2)\,dy\,dx = 3/7$$

(10.d)



$$V = \int_{-a/2}^{a/2}\int_0^{\sqrt{a^2-4x^2}}(my)\,dy\,dx = \frac{ma^3}{3}$$

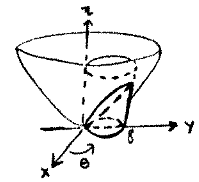
(10e)



$$V = \int_{-4}^1\int_{3x}^{4-x^2}(x+4)\,dy\,dx = \int_{-4}^1 xy+4y\Big|_{3x}^{4-x^2}dx$$

$$= \int_{-4}^1 x(4-x^2)+4(4-x^2)-\left[x(3x)+4(3x)\right]dx = \frac{625}{12}$$

(11.)



$x^2+y^2=4z$
$r^2=4z$
$z=r^2/4$

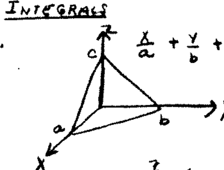
$x^2+y^2=8y$
$r^2=8r\sin\theta$
$r=8\sin\theta$

$$V = \iint z\,dA = \int_0^\pi\int_0^{8\sin\theta} z\,r\,dr\,d\theta = \int_0^\pi\int_0^{8\sin\theta}\frac{r^3}{4}\,dr\,d\theta = 256\int_0^\pi\sin^4\theta\,d\theta$$

$$= 96\pi$$

TRIPLE INTEGRALS

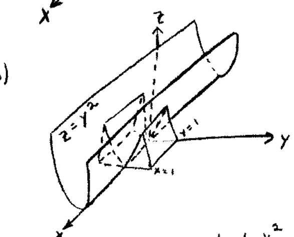
(1) a. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$



$$V = \int_0^a\int_0^{b(1-x/a)}\int_0^{c(1-x/a-y/b)}dz\,dy\,dx$$

$$= \int_0^a cy-\frac{c}{a}xy-\frac{c}{b}\frac{y^2}{2}\Big|_0^{\frac{b(a-x)}{a}}dx = \frac{abc}{6}$$

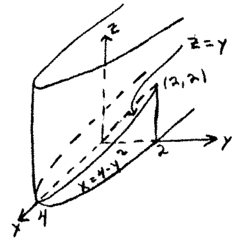
(b)



$$\tfrac12 V = \iiint dV = \int_0^1\int_0^1\int_0^{y^2}dz\,dy\,dx = 1/3$$

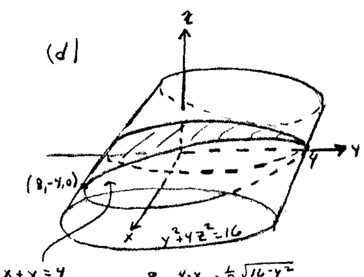
$$V = 2/3$$

(c)



$$V = \int_0^2\int_0^{4-y^2}\int_0^y dz\,dx\,dy = 4$$

(d)

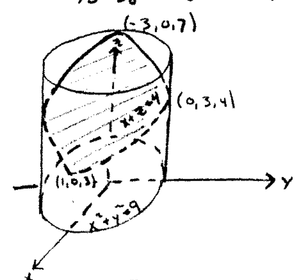


$$V = \int_0^8\int_{-4}^{4-x}\int_{-\frac12\sqrt{16-y^2}}^{\frac12\sqrt{16-y^2}}dz\,dy\,dx$$

(easier integration) $V = \int_{-4}^4\int_{-\frac{\sqrt{16-y^2}}{2}}^{\frac{\sqrt{16-y^2}}{2}}\int_0^{4-y}dx\,dz\,dy$

$$= 32\pi$$

(e)



$$V = \int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{4-x}dz\,dy\,dx$$

$$= \int_{-3}^3\left[8\sqrt{9-x^2}-2x\sqrt{9-x^2}\right]dx$$

$$= 8\left\{\frac92\sin^{-1}\frac{x}{3}+\frac{x\sqrt{9-x^2}}{2}\right\}_{-3}^{3} = 36\pi$$