

Mathematics Support Capsules

Applications of the Derivative:
MAX-MIN PROBLEMS
by Ann Michel

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Max-min problems are word problems in which you are asked to maximize or minimize a certain quantity. They are sometimes called optimization problems.

A strategy for solving these kinds of problems will be outlined here, but the only way to become adept at solving max-min problems is to do lots of them. So give yourself 45 minutes to an hour to sit down and work through some problems.

Pre-requisites:

- how to take first derivatives
- how the 1st derivative relates to the maximum & minimum points of a function. (If you are at all shaky on this, see module GRAPHING WITH CALCULUS.)
- formulas for perimeter, volume & area for various shapes. (See Capsule on GENERAL FORMULAE.)

Example 1

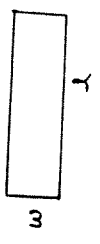
What rectangle with a Perimeter of 12 feet has the greatest area?

1 Draw a picture:



2 Invent Symbols:

w = width
ℓ = length



3 Set up Equations:

Look back at the problem to determine which equations you need.
Note the key words: perimeter and area.

$$P = 2w + 2\ell$$

$$A = w\ell$$

Use the given information $P = 12$:

$$12 = 2w + 2\ell$$

4. Determine the Quantity to be maximized or minimized. Here, we are looking for the greatest Area:

$$A = w\ell$$

5. Express this quantity in terms of One Variable. Here's where we use the other equation, $12 = 2w + 2\ell$

Solve for one of the two variables, say ℓ :

$$12 = 2w + 2\ell$$

$$12 - 2w = 2\ell$$

$$6 - w = \ell$$

so now express the area as a function with one variable

$$A = (6-w)w \quad A = 6w - w^2$$

6. Take the Derivative

$$\frac{dA}{dw} = 6 - 2w$$

A may be a max when $\frac{dA}{dw} = 0$ or when $\frac{dA}{dw}$ does not exist.

$$0 = 6 - 2w$$

$$6 = 2w$$

$$w = 3$$

$w = 3$ is the critical value for the function $A = 6w - w^2$

7. Determine the Interval of Possible values for w .

the width w must be within the interval

$$0 < w < 6 \quad (\text{if the perimeter} = 12, \text{ the width cannot be } > 6)$$

8. Test endpoints and critical points for max A.

So check the function $A = 6w - w^2$ for the points $w=0, 6$ -endpoints
3-critical points.

for $w=0$ $A = 6(0) - 0^2 = 0$
for $w=6$ $A = 6(6) - 6^2 = 0$
for $w=3$ $A = 6(3) - 3^2 = 9$ ← this is the maximum area, occurring for $w=3$.

So the rectangle with the greatest area, has width = 3.

To solve for length, use the formula

$$p = 2w + 2\ell$$

$$12 = 2(3) + 2\ell$$

$$6 = 2\ell$$

$$3 = \ell$$

The largest rectangle is a square, with side = 3.

9. This answer looks Reasonable.

Strategy

0. Read the problem carefully, several times. Many errors are due to careless reading & subsequent misinterpretation of the question.
1. Draw a diagram. This is an excellent way to get you thinking mathematically; away from English words & into symbols.
2. Invent Symbols for the quantities involved. P for perimeter, etc.
3. Set up an Equation, or set of equations. These almost always are about perimeter, area, volume, or cost.
4. Determine the Quantity to be maximized or minimized.
5. Express this quantity in terms of One Variable.
6. Take the Derivative & determine the critical value(s) (set derivative = 0, and solve for the variable, also list values of the variable for which the derivative does not exist.)
7. Determine the Interval of acceptable values for your (one) variable.
8. Test the endpoints of the interval and the critical points in the formula from step 5 to determine the absolute min or the absolute max. (There is a theorem - see your calculus text - that the absolute max or min must occur at a critical point or at the endpoints of the interval).
Note: Do Not omit finding the endpoints and testing them. Professors love to set traps for this.
9. Then look at your answer. Reread the question. Does your answer seem Reasonable?

Strategy Summary: Read, Draw, Symbols, Equation, Quantity, One Variable, Derivative, Interval, Test, Reasonable?

Example 2:

You are asked to design a tin drum to hold 100 cubic inches of chemical, using the least amount of material possible.



1. Draw:

2. Symbols:

r = radius of (circular) base
 h = height

3. Equations:

We know the volume:

$$V = \pi r^2 h \quad V = 100 \text{ (given)} \quad 100 = \pi r^2 h$$

Trick coming up:

We are going to need two equations, since we have 2 unknowns: r and h . So we need to come up with some other equation relating these two quantities.

4. The Quantity to be minimized is the amount of material, the surface area. So for the second equation, use S.A. of a cylinder = $2\pi r^2 + 2\pi r h$

5. Express this area in terms of One Variable.

$$100 = \pi r^2 h$$

$$h = \frac{100}{\pi r^2}$$

$$\text{S.A.} = 2\pi r^2 + 2\pi r \left(\frac{100}{\pi r^2} \right)$$

$$\text{S.A.} = 2\pi r^2 + \frac{200}{r}$$

6. Derivative:

$$\frac{d(\text{S.A.})}{dr} = 4\pi r - \frac{200}{r^2}$$

$$\text{minimize: } 4\pi r - \frac{200}{r^2} = 0$$

$$4\pi r = \frac{200}{r^2}$$

$$r^3 = \frac{50}{\pi}$$

$$r = \sqrt[3]{\frac{50}{\pi}}$$

critical point

7. Determine endpoints of Interval:

$$0 \leq r \leq \infty$$

It must be a positive number, but (theoretically) the radius could extend to ∞ (imagine an incredibly flat huge disk)

8. Test possible points:

$$\text{Plug in } \text{S.A.} = 2\pi r^2 + \frac{200}{r}$$

$$r = 0 \quad \text{S.A. is undefined } \left(\frac{200}{r} \right)$$

$$r \rightarrow \infty \quad \text{S.A.} \rightarrow \infty$$

$$r = \sqrt[3]{\frac{50\pi}{\pi}} \quad \text{S.A.} = 2\pi \left(\frac{50}{\pi} \right)^{2/3} + \frac{200}{\sqrt[3]{\frac{50\pi}{\pi}}}$$

This is the minimum quantity.
 $\left(\sqrt[3]{\frac{50}{\pi}} = 4 \right)$

Minimum dimensions:

$$r = \sqrt[3]{\frac{50}{\pi}}$$

$$V = \pi r^2 h$$

$$V = \pi \left(\frac{50}{\pi} \right)^{2/3} h$$

$$\frac{100}{\pi} = \pi^{1/3} \left(\frac{50}{\pi} \right)^{2/3} h$$

$$h = \frac{100}{\sqrt[3]{2500\pi}}$$

9. Answer is Reasonable.

Example 3.

Find two positive numbers whose sum is a , such that the product of the cube of one number and the square of the other is a maximum.

1. Draw: Not possible here.

2. Symbols: 2 numbers - call one x
and the other $a - x$
(sum = a , $x + (a - x) = a$)

3. Equations: Note: There is only one variable x , so only one equation is needed.

$$\text{Product} = x^3(a-x)^2$$

4. Quantity: We are maximizing the product.

5. One Variable: Done

$$\begin{aligned} \text{Product} &= x^3(a^2 - 2ax + x^2) \\ &= x^3a^2 - 2ax^4 + x^5 \end{aligned}$$

6. Derivative: $\frac{d(\text{Product})}{dx} = 3x^2a^2 - 8ax^3 + 5x^4$

$$0 = 3x^2a^2 - 8ax^3 + 5x^4$$

$$0 = x^2(3a^2 - 8ax + 5x^2)$$

$$0 = x^2(3a - 5x)(a - x)$$

$$\text{critical points: } x = 0, x = a, x = \frac{3a}{5}$$

7. Interval: $0 \leq x \leq a$ Each number is positive (≥ 0) and less than sum (= a).

8. Test:

$$x = 0 \quad \text{Product} = 0$$

$$x = a \quad \text{Product} = a(a-a)^2 = 0$$

$$x = \frac{3a}{5} \quad \text{Product} = \left(\frac{3a}{5}\right)^3 \left(a - \frac{3a}{5}\right)^2 = \frac{81a^3}{125} \left(\frac{4a}{5}\right)^2 \leftarrow \text{maximum}$$

$$\text{one number} = x = 3^3 \cdot 2^2 \cdot \left(\frac{a}{5}\right)^5$$

$$\text{other number} = a - 3^3 \cdot 2^2 \cdot \left(\frac{a}{5}\right)^5$$

9. Reasonable: Yes, answer checks.



For the next 5 problems,
solutions are presented in 4 parts.
Try and do as much as possible on your own.



Refer to the partial solutions only if you are really stuck.

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1. Postal regulations state that a parcel must have combined length and girth of at most 72". What dimensions should be used to construct a box with a square cross section which will comply with regulations, and have the most volume?
 2. A rectangular box with a square bottom and no top is to have a volume of 6 cu.ft. The material for the bottom costs \$3/sq.ft. and the material for the sides costs \$2/sq.ft. What dimensions should the box have to minimize costs?
 3. Find the rectangle of greatest area that can be inscribed in a given circle.

4. A local cable TV company wants to lay cables from its headquarters, across a river 900 ft. wide to a new home 3000 ft. downstream, on the opposite bank of the river. Running cable under water costs \$5/ft, while overland costs run \$4/ft. Describe the most economical route to lay the cable.
- ★ 5. (Trickier) What is the maximum volume of the box formed when four equal squares are cut from the corners of a 16' × 20' piece of cardboard?

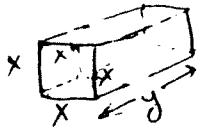
Partial Solution 1:

1. The word 'girth' might be confusing here.

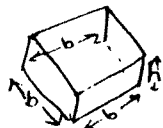


Imagine a rubber band wrapped around the box.
the length of the band is the length of the girth.

Draw a picture:



2. Draw

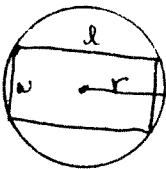


h = height
 b = bottom side length

What quantity are you minimizing?

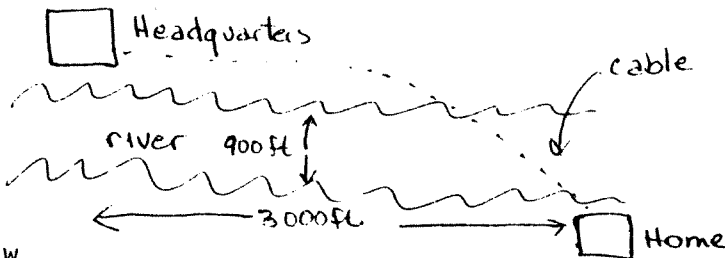
3.

Draw; Invent symbols

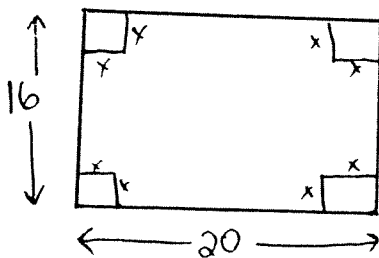


r = radius of circle
 l = length of rectangle
 w = width of rectangle

4. Draw



5. Draw



Partial Solution 2:

1. Symbols: $4x =$ girth
 $y =$ length

Equation: $y+4x = 72$ "parcel must have combined length & girth of at most 72"

Quantity to maximize: Volume

$$V = (\text{length})(\text{height})(\text{width})$$

$$V = yxx$$

2. Equations:

$$V = hb^2 = 6$$

Realize here that the quantity to be minimized is the cost of the material.

cost of bottom = \$3 sq. ft. dimensions of bottom = b^2

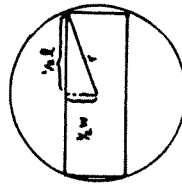
cost of sides = \$2 sq. ft. dimensions of sides = $4(hb)$

(four small rectangles for the four sides.)

your answer will be in dollars

3. Equations Area of rectangle = $A = wl$

To get the second equation is a little trickier. We need to maximize the Area, but it is in 2 variables. So, we need some way to relate the dimensions of the circle to the dimensions of the rectangle.
 Observe:

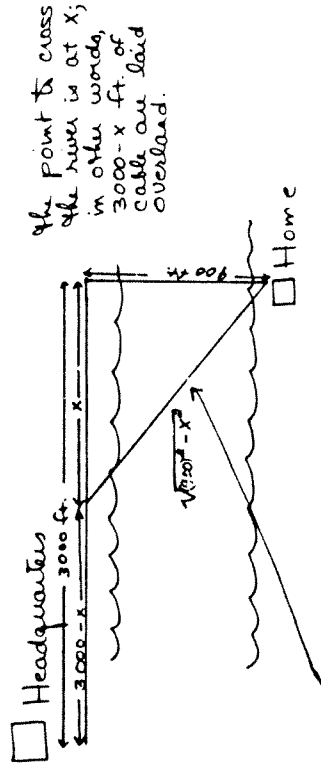


using the Pythagorean theorem

$$\left(\frac{1}{2}x\right)^2 + \left(\frac{1}{2}y\right)^2 = r^2$$

$$\frac{1}{4}x^2 + \frac{1}{4}y^2 = r^2$$

4. You are finding a route to lay the cable; specifically, exactly where does the cable cross the river & have to be laid at the higher cost?



To find this distance, apply the Pythagorean Theorem

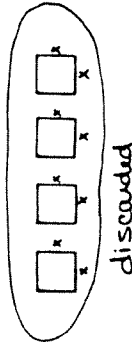
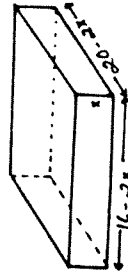
$$(\text{Underwater distance})^2 = 900^2 + x^2$$

$$\text{Underwater distance} = \sqrt{900^2 + x^2}$$

5. Area = $lw = (16)(20) = 320$

Dimensions after cutting: $l = 20 - 2x$ $w = 16 - 2x$

Volume = $(20-2x)(16-2x)x$



MM-10

Partial solution 3:

1. Express in one variable only

$$72 = y + 4x, y = 72 - 4x$$

$$V = yx^2 = (72 - 4x)x^2 = 72x^2 - 4x^3$$

Take the derivative: $\frac{dV}{dx} = 144x - 12x^2$

$$\text{Max at } \frac{dV}{dx} = 0 \quad 144x - 12x^2 = 0 \quad 144x = 12x^2$$

$$12x = x^2 \quad ; \quad x = 12, x = 0$$

2. Cost = $b^2(3 \text{ dollars}) + 4bh$ (2 dollars)

$$c = 3b^2 + 8bh$$

$$V = hb^2 = 6 \quad h = 6/b^2$$

$$c = 3b^2 + 8b(6/b^2)$$

$$c = 3b^2 + \frac{48}{b}$$

Derivative: $\frac{dc}{db} = 6b - \frac{48}{b^2} = 0$

3. Express in terms of one variable, say l :

$$l^2 = 4r^2 - \omega^2$$

$$l = \sqrt{4r^2 - \omega^2}$$

$$A = \omega \sqrt{4r^2 - \omega^2}$$

Take the Derivative (treat r as a constant)

$$\frac{dA}{d\omega} = \sqrt{4r^2 - \omega^2} + \frac{\omega}{2} \frac{(-2\omega)}{\sqrt{4r^2 - \omega^2}} = \frac{4r^2 - \omega^2 - \omega^2}{\sqrt{4r^2 - \omega^2}}$$

4. Equations - remember, you are trying (as in problem 2) to minimize cost

$$\text{cost} = \$4 \text{ (overland cable)} + \$5 \text{ (underwater cable)}$$

$$\text{cost} = \$4 (3000-x) + \$5 (\sqrt{900^2 + x^2})$$

5. Note - we don't really need the area, since the problem has only one variable, x .

$$\text{Maximize } \frac{dV}{dx} = 320 - 144x + 12x^2$$

Final Answer:

1. Interval for x : $0 \leq x \leq \frac{72}{4}$ max. girth
 Plug in values $x=0$ $x=12$, $x = \frac{72}{4}$ to $V = 72x^2 - 4x^3$
 $x = 0$ $V = 0$ cubic inches
 $x = 12$ $V = 3,456$ cubic inches ← max. volume
 $x = \frac{72}{4} = 18$ $V = 2,000$ cubic inches
 dimensions: $x = 12''$
 $72'' = 4(12)'' + y$ $y = 24''$
 box = $12'' \times 12'' \times 24''$

2. $b = 2$
 Interval $0 \leq b \leq \infty$
 $b = 0$ $c = \text{undefined}$
 $b = \sqrt{6}$ $c = 108 + \frac{48}{\sqrt{6}}$
 $b = 2$ $c = 12 + 24 = 36$ + minimum

Minimum costs occur when the bottom is $2' \times 2'$ and the height is $6 = hb^2$ $6 = h4$ $h = (\frac{3}{2})'$

3. $\frac{dA}{dw} = \frac{4r^2 - 2w^2}{\sqrt{4r^2 - w^2}} = 0$ when $w = \sqrt{2}r$

Interval: $0 \leq w \leq 2r$ (width can't be wider than the circle diameter)

when $w = \sqrt{2}r$
 $A = 0$
 $w = 2r$ $A = 0$
 $w = \sqrt{2}r$ $A = 2r^2$
 $l^2 = 4r^2 - (\sqrt{2}r)^2$
 $l^2 = 2r^2$ $l = \sqrt{2}r$

4. Derivative

$\frac{d(\text{Cost})}{dx} = \frac{5x}{\sqrt{900^2 + x^2}} - 4 = 0$
 $\sqrt{900^2 + x^2} = \frac{5}{4}x$
 $900^2 + x^2 = \frac{25}{16}x^2$
 $x^2 = \frac{19}{6}(900)^2$
 $x = 1200$

Interval

- $0 \leq x \leq 3000$
 $x = 0$ cost = 16,500
 $x = 3,000$ cost = > 15,000
 $x = 1,200$ cost = 14,700 + min

the route is: go overland (3000-x) meters or 1800 meters, & then cross underwater to the home.

5. $\frac{dV}{dx} = 0$

$4(3x^2 - 36x + 80) = 0$

doesn't factor

$x = 36 \pm \sqrt{\frac{536}{6}}$

$x = 6 \pm \frac{2}{3}\sqrt{21} = 6 \pm 3\frac{1}{2} = 9\frac{1}{2}, 2\frac{1}{2}$

Interval $0 \leq x \leq 8$ + since you are taking $2x$ out of the side whose length is 16.

NOTE the value $9\frac{1}{2}$ is out of the interval! Do not even consider it!

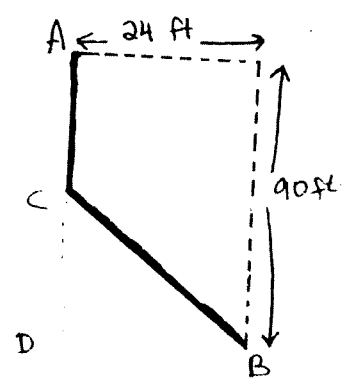
$x = 0$ $V = 0$
 $x = 8$ $V = 0$
 $x = 3/2$ $V = \text{max!}$

so cut $3/2''$ squares out of the corners.

More Problems

6. During the summer, members of a scout troupe have been collecting used bottles that they plan to deliver to a glass company for recycling. So far, in 80 days the scouts have collected 24,000 kilograms of glass for which the glass company currently offers 1 cent per kilogram. However, because bottles are accumulating faster than they can be recycled, the company plans to reduce by 1 cent each day the price it will pay for 100 kilograms of used glass. Assume that the scouts can continue to collect bottles at the same rate and that transportation costs make more than one trip to the glass company unfeasible. What is the most profitable time for the scouts to conclude their summer project and deliver the bottles?
7. It is noon, and the hero of a popular spy story is driving a Jeep through the sandy desert in the tiny principality of Alta Loma. He is 32 kilometers from the nearest point on a straight, paved road. Down the road 16 kilometers is a power plant in which a band of international saboteurs has placed a time bomb set to explode at 12:50 p.m. The Jeep can travel 48 kilometers per hour in the sand and 80 kilometers per hour on the paved road. If he arrives at the power station in the shortest possible time, how long will our hero have to defuse the bomb?
8. The sum of 2 positive numbers is 16. Find the two numbers if the product of their squares is a max.
9. Find the positive real number between 0 and 1 whose square exceeds its cube by the greatest value .
10. Assuming that the strength of a beam of rectangular cross-section is directly proportional to the width and to the cube of the altitude, find the width of a beam of maximum strength that may be cut out of a log of diam. 16 cm.

11. A ditch is to be dug connecting A and B. The earth along AD is soft, but that to the right of AD is hard. The cost of digging the portion AC is \$10/ft. and the cost of digging the portion CB is \$20/ft. Where should the turn C be made for a minimum cost?



12. If a ball is thrown vertically upward with a velocity of 32 ft/sec, its height after t sec is given by the equation

$$s = 32t - 16t^2.$$

At what instant will the ball be at its highest point, and how high will it rise?

13. A chemist makes 100 ounces of perfume. He knows that at a price of \$7.00 per ounce he can sell all the perfume. However, for each \$.50 increase in price per ounce, one less ounce is sold. For each ounce he does not sell, he saves \$.02 in bookkeeping costs. What price should he charge for maximum income?

Answers and Solutions to Additional Problems:

6. Scout's profits = (total # bottles collected) × (what company will pay per bottle)

$$P = (24,000 + 300t) \times \left(\frac{100-t}{100}\right)$$

$$= 24,000 + 60t - 3t^2$$

(quadratic!)

$$\frac{dP}{dt} = 60 - 6t = 0 \text{ when } t = 10$$

let $t =$ #days from now

$$\frac{24,000}{80} = \frac{300 \text{ kg}}{\text{day}} =$$

rate at which bottles are collected

now company pays

1¢/kg or 100¢/100 kg

company lowers price

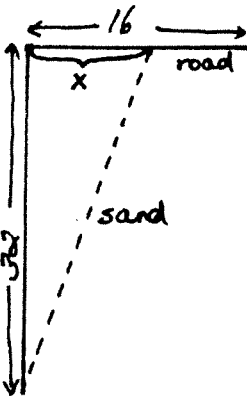
$\frac{t}{100}$ per day

t	P
10	\$243.00
0	\$240.00

max profit

EGAD!
10 more days of collecting isn't worth \$3. Hope the scouts know enough graphing to notice this.

7.



We want to minimize

time $T =$ time on sand + time on road

$$T = \frac{\text{distance on sand}}{\text{rate on sand}} + \frac{\text{distance on road}}{\text{rate on road}}$$

$$T = \frac{\sqrt{x^2 + (32)^2}}{48} + \frac{16-x}{80}$$

$$\frac{dT}{dx} = \frac{2x}{(2)48\sqrt{x^2 + (32)^2}} - \frac{1}{80} = 0$$

when $80x = 48\sqrt{x^2 + (32)^2}$

$$\frac{5}{3}x = \sqrt{x^2 + (32)^2}$$

$$\frac{25}{9}x^2 = x^2 + (32)^2$$

$$\frac{16}{9}x^2 = (32)^2 \text{ so}$$

$$x^2 = \frac{(32)^2(9)}{16}$$

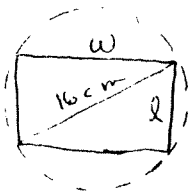
So-min. time

must be at $x=0$ or at $x=16$, "endpts"
 $T = .86$ $T = .74$

12:50 is .83 hrs. after noon;
 $.83 - .74 = .09 \overset{\text{hrs.}}{\Delta} = 5.4 \text{ min. to defuse bomb.}$

and taking sq root both sides gives $x=24$, which is not even within $0 \leq x \leq 16$!

10.



$$\text{strength} = k\omega h^3$$

$$h^2 + \omega^2 = 16$$

$$= k\sqrt{16-h^2} h^3$$

max strength

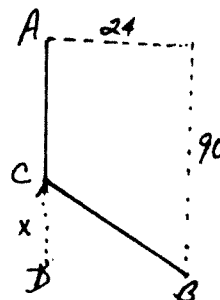
at $\omega=8$

11.

$$x = CD$$

$$\text{Cost} = 20\sqrt{x^2 + 24^2} + 10(90-x)$$

minimized for $x \approx 14$ feet



12. $s = 32t - 16t^2$ so $v = 32 - 32t$

$V=0$ when $t=1$ at $t=1$, $s=16$ ft.

13. $x = \#$ of .50 increments in price

$$\text{Income} = (7 + .50x)(100-x) + .02x = 700 + 43.02x - .50x^2$$

$$\frac{dI}{dx} = 43.02 - x = 0 \text{ when } x = 43.02, \text{ which is a maximum}$$

because $\frac{d^2I}{dx^2} < 0$.

so price he should charge is $\$7 + (43.02) \times \$.50 = \$28.51$? EGAD!