MATH SUPPORT CAPSULE: INFINITE SERIES, CONVERGENCE

Infinite Series I: Convergence and Divergence.

In section E of this capsule you will find 22 problems (with solutions worked out) concerning the convergence of infinite series. So if you want to practice these techniques or test your current understanding of them, you may want to turn directly to section E.

In sections A through D, we present a brief review of the most important tests for convergence, and several examples. These should prove helpful in solving most of the problems in section E.

A. Geometric Series

A geometric series is a series of the form

$$\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + \dots .$$

This series diverges if $|r| \geq 1$. It converges if $|r| < 1$, in which case

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r} .$$

Example 1. The series $1-2+4-8+\dots+(-2)^i+\dots$ diverges, because $|r| = 2 \geq 1$.

Example 2. The series

$$2+1+\tfrac{1}{2}+\tfrac{1}{4}+\dots = \sum_{i=0}^{\infty} 2\cdot\left(\tfrac{1}{2}\right)^i$$

converges to $\dfrac{2}{1-\frac{1}{2}} = 4$, because $r = \frac{1}{2}$.

B. A Test Which Applies to All Series

Let $\sum_{i=1}^{\infty} a_i$ be any series. For there to be any possibility that $\sum_{i=1}^{\infty} a_i$ converges, we must have $\lim_{i\to\infty} a_i = 0$. If this limit does not exist, or exists but does not equal zero, then $\sum_{i=1}^{\infty} a_i$ diverges. This test is often easier to apply than the others, and is useful for weeding out the worst divergent series. However, it cannot be used to prove convergence, only divergence. To sum up:

> If $\lim_{i\to\infty} a_i$ does not exist or if $\lim_{i\to\infty} a_i \neq 0$, then $\sum_{i=1}^{\infty} a_i$ diverges.

Example 3. Does the series $\sum_{n=1}^{\infty} (\sqrt{n^2+n} - n)$ converge or diverge?

Solution: We are given $a_n = \sqrt{n^2+n} - n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{n^2+n} - n = \lim_{n\to\infty} (\sqrt{n^2+n} - n)\frac{(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \neq 0.$$

So the series diverges.

C. Positive Series

The following tests can be tried for any series all of whose terms are positive.

1. Integral Test.

Say we are given a series, $\sum_{i=1}^{\infty} a_i$. Suppose a_i is given as some function of i, $a_i = f(i)$, which is defined not only for integers, but for all positive real numbers i. Finally, suppose f is a positive, decreasing function. Then the series $\sum_{i=1}^{\infty} a_i$ and the integral $\int_1^{\infty} f(t)dt$ either both converge or both diverge. Note that the particular lower limit of integration we choose makes no difference in this test.

Example 4. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

Solution: The reader should check that the function $f(x) = \frac{\ln x}{x}$ is indeed positive and decreasing when $x \in [3,\infty)$. So we compute:

$$\int_3^{\infty} \frac{\ln t}{t}\, dt = \int_3^{\infty} \ln t \; d(\ln t) = \frac{1}{2}(\ln t)^2\Big|_3^{\infty}$$

$$= \lim_{N\to\infty} \frac{1}{2}((\ln N)^2 - (\ln 3)^2)$$

which is infinite; so the integral, and thus the series, diverge.

2. The p-series.

Let $p > 0$ be a real number. Then the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

(a) converges if $p > 1$

(b) diverges if $0 < p \leq 1$.

Thus $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$ $(p = 1)$ diverges, while $1+\frac{1}{2^2}+\frac{1}{3^2}+\dots$ $(p = 2)$ converges.

These facts are easily proven from the integral test.

3. Ratio Test

Consider the series $\sum_{i=1}^{\infty} b_i$ where $b_i \geq 0$ for all i. Suppose the limit

$$L = \lim_{i\to\infty} \frac{b_{i+1}}{b_i} \text{ exists (and is finite).}$$

Then if $L < 1$, the series converges. If $L > 1$, the series diverges. If $L = 1$, or if the limit doesn't exist, then the ratio test doesn't tell us whether the series converges.

Example 5. $\sum_{n=0}^{\infty} \frac{1}{n!}$

Solution:

$$L = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)\cdot n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So the ratio test tells us this series converges.

4. Comparison Tests

We are still working only with positive series. Here are some tests enabling us to deduce the convergence or divergence of one series by comparing it to some other, more well understood, series.

(a) Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series, with $0 \leq a_i \leq b_i$ for all i. Then.

$$\sum a_i \text{ diverges} \Longrightarrow \sum b_i \text{ diverges}$$

while

$$\sum b_i \text{ converges} \Longrightarrow \sum a_i \text{ converges.}$$

Remark 1. We can deduce nothing if we know only that $\sum a_i$ converges, or that $\sum b_i$ diverges.

Remark 2. Before applying this (or any other) test for convergence we may, if we wish, drop any finite number of terms from the series, since this does not alter the convergence or divergence of a series.

Example 6. $\sum_{n=2}^{\infty} \frac{\ln n}{n - \frac{1}{2}}$

Solution: We showed in example 4 that $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ diverges. Since

$$\frac{\ln n}{n} \leq \frac{\ln n}{n - \frac{1}{2}} \quad \text{for } n = 2,3,4,\ldots,$$

we deduce from the comparison test that $\sum_{n=2}^{\infty} \frac{\ln n}{n - \frac{1}{2}}$ diverges also.

Example 7. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Solution: We know from the p-series that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Since

$$\frac{1}{n^2+1} < \frac{1}{n^2} \quad \text{for } n = 1,2,3,\ldots,$$

we deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges as well.

(b) Ratio Comparison Test

Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two positive series, and suppose the limit

$$L = \lim_{i\to\infty} \frac{a_i}{b_i}$$

exists and is nonzero. Then the two series either both converge or they both diverge. In other words, if we know one series converges (or diverges), then so does the other.

D. Non-positive Series

We now consider series not all of whose terms are positive.

1. Alternating series. If the sign of the terms of a series alternates between plus and minus, the series is called an alternating series.

An alternating series converges if the following two conditions are both satisfied:

(a) $|a_{i+1}| \leq |a_i|$ for all i

and (b) $\lim_{i\to\infty} a_i = 0$.

2. Absolute Convergence. Say $\sum a_i$ is a series not all of whose terms are positive. If it is not alternating, or if it alternates but does not satisfy the conditions in the previous paragraph, there is one more test to try. The series is said to be absolutely convergent if the positive series $\sum_i |a_i|$ converges. Being positive, this last series can be tested by the several methods given previously. Finally, we use the fact that every absolutely convergent series converges.

Remark: If we find that $\sum |a_i|$ diverges, we cannot conclude that $\sum a_i$ diverges. For example, the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

diverges while

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges by the alternating series paragraph.

E. **Problems**

Figure out whether or not each of the following series converges: (A star indicates a harder than average problem.)

1. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

2. $\sum_{n=1}^{\infty} \frac{n!}{2^{(n^2)}}$

3. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

4. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{3/2}}$

5. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

6. $\sum_{n=1}^{\infty} \frac{(2^n)^2}{n!}$

*7. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)$

8. $\sum_{n=1}^{\infty} \frac{n! n^{10}}{(2n)!}$

9. $\sum_{n=1}^{\infty} (\frac{\ln n}{n})^2$

*10. $\sum_{n=1}^{\infty} (\sqrt[(n^2)]{n} - 1)$

11. For what positive values of a and b does this series converge:

$$\sum_{n=1}^{\infty} \frac{n^a}{b^n}$$

12 and 13. For what positive values of a do these series converge:

12. $\sum_{n=1}^{\infty} \frac{n^a}{n!}$

13. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$

*14. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

15. $\sum_{n=1}^{\infty} (\frac{\ln n}{n})^n$

16. $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{n}$

17. $\sum_{n=1}^{\infty} \frac{(2n)!}{n^n}$

18. $\sum_{n=1}^{\infty} \frac{n!}{n^{\frac{1}{2}n}}$

19. For what positive values of a does the series $\sum_{n=1}^{\infty} \frac{n!}{n^{an}}$ converge?

*20. $\sum_{n=1}^{\infty} (-1)^n (\frac{2-\cos n\pi}{n})$

21. $\sum_{n=1}^{\infty} \cot^{-1}(n)$

22. $\sum_{n=1}^{\infty} \frac{(n!)^2 2^n}{(2n)!}$

Solutions to Problems

We only give one way of deciding whether each of these series' converges. Since there are usually many ways of deciding convergence, your answer needn't match ours completely to be correct. However, your final conclusion should be the same!

1. $a_n = \frac{n!}{2^n}$

First, note that $a_1 = \frac{1}{2}$. Also, note that for each $n \geq 1$, $a_{n+1} = \frac{n+1}{2} \cdot a_n$. Thus, a_n is actually an increasing function of n. So $a_n \geq \frac{1}{2}$ for all n. It follows that $\lim_{n\to\infty} a_n \neq 0$. So the series diverges.

2. $a_n = \frac{n!}{2^{(n^2)}}$ Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n! \cdot 2^{n^2}}{2^{n^2+2n+1} \cdot n!} = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} = 0.$$

So the series converges.

3. $a_n = \frac{1}{n \ln n}$

Since $f(x) = \frac{1}{x \ln x}$ is a positive decreasing function, we use the integral test:

$$\int_2^\infty \frac{dx}{x \ln x} = \int_2^\infty \frac{d(\ln x)}{\ln x}$$

$$= \lim_{N \to \infty} \Big[\ln(\ln x)\Big]_2^N = \lim_{N \to \infty} \Big[\ln(\ln N) - \ln(\ln 2)\Big]$$

$= \infty$. So the integral and thus the series diverge.

4. $a_n = \frac{1}{n (\ln n)^{3/2}}$ Integral test:

$$\int_2^\infty \frac{dx}{x(\ln x)^{3/2}} = \int_2^\infty \frac{d(\ln x)}{(\ln x)^{3/2}}$$

$$= \lim_{N \to \infty}\Big[-2(\ln x)^{-\frac{1}{2}}\Big]_2^N = \lim_{N \to \infty} \frac{2}{\sqrt{\ln 2}} - \frac{2}{\sqrt{\ln N}} =$$

$\frac{2}{\sqrt{\ln 2}}$. So the series converges since the integral does.

5. $a_n = \frac{(-1)^n}{\ln n}$

In this alternating series, we have $|a_{n+1}| < |a_n|$ for all n, and

$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{(-1)^n}{\ln n} = 0$. By the alternating series test, it converges.

6. $a_n = \frac{(2^n)^2}{n!}$ Ratio test:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(2^{n+1})^2}{(n+1)!} \cdot \frac{n!}{(2^n)^2}$$

$$= \lim_{n \to \infty} \frac{2^{2n+2} \cdot n!}{(n+1) \cdot n! \cdot 2^{2n}} = \lim_{n \to \infty} \frac{4}{n+1} = 0.$$

So the series converges.

* 7. $a_n = \sqrt[n]{n} - 1$

First, consider the series $\sum_{n=1}^\infty \frac{\ln n}{n}$.

Since this series is greater, term by term, than the divergent series $\sum \frac{1}{n}$, it diverges by the comparison test.

Now we use the ratio comparison test, with $a_n = \sqrt[n]{n} - 1 = n^{1/n} - 1$ and $b_n = \frac{\ln n}{n}$.

We compute:

$$\frac{a_n}{b_n} = \frac{n^{1/n} - 1}{\ln n / n} = \frac{e^{\ln(n^{1/n})} - 1}{\ln n / n}$$

$$= \frac{e^{\frac{\ln n}{n}} - 1}{\ln n / n}.$$ In the above, we used the fact that $e^{\ln a} = a$ for all a, and that $\ln a^x = x \ln a$ for all a, x.

Now, let

$$L = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{e^{\ln n / n} - 1}{\ln n / n}.$$

Using l'Hôpital's rule (replacing numerator and denominator by their derivatives), we obtain:

$$L = \lim_{n \to \infty} \frac{e^{\ln n/n} \cdot D_n\left(\frac{\ln n}{n}\right)}{D_n\left(\frac{\ln n}{n}\right)}$$

where 'D_n' indicates differentiation with respect to n. So

$$L = \lim_{n \to \infty} e^{\ln n / n} = e^0 = 1.$$ Since Σb_n diverges, so does Σa_n, by the ratio comparison test.

8. $a_n = \frac{n!\, n^{10}}{(2n)!}$ Ratio Test:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(n+1)!\,(n+1)^{10}}{(2n+2)!} \cdot \frac{(2n)!}{n!\, n^{10}}$$

$$= \lim_{n \to \infty} \frac{(n+1)\, n!\, (2n)!}{(2n+2)(2n+1)(2n)!\, n!} \cdot \left(\frac{n+1}{n}\right)^{10}$$

$$= \lim_{n \to \infty} \frac{n+1}{(2n+2)(2n+1)} \cdot \left(1 + \frac{1}{n}\right)^{10} = 0.$$ So the series converges.

9. $a_n = \left(\frac{\ln n}{n}\right)^2$ Integral test:

$$\int_2^\infty \frac{(\ln x)^2}{x^2}\, dx.$$

$$\boxed{\begin{array}{ll} \text{Substitute} & \\ u = \ln x & du = \frac{dx}{x} \\ x = e^u & \end{array}}$$

Obtain, after subst.,

$$\int_{\ln 2}^\infty \frac{u^2}{e^u}\, du = \int_{\ln 2}^\infty u^2 e^{-u}\, du.$$

We leave it to the reader to compute this integral, showing it converges. (Hint: Use integration by parts.) Thus the series converges.

* 10. $a_n = \sqrt[(n^2)]{n} - 1 = n^{\frac{1}{n^2}} - 1.$

Similar to problem 7. Consider the series $\sum \frac{\ln n}{n^2}$. It converges because it is smaller, term by term, than the convergent series of problem 9.

Now use the ratio comparison test, with $a_n = n^{1/n^2} - 1$ and $b_n = \frac{1}{n^2}\ln n$.

As in problem 7,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{\frac{1}{n^2}\ln n} - 1}{\frac{1}{n^2}\ln n}$$

$$\underset{\text{L'Hôpital's rule)}}{=} \lim_{n\to\infty} \frac{e^{\frac{1}{n^2}\ln n} \cdot D_n\left(\frac{1}{n^2}\ln n\right)}{D_n\left(\frac{1}{n^2}\ln n\right)}$$

$= \lim_{n\to\infty} e^{\frac{1}{n^2}\ln n} = 1.$ So the series converges by the ratio comparison test.

11. $a_n = \frac{n^a}{b^n}$ $\quad a, b > 0 \quad$ Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^a}{b^{n+1}} \cdot \frac{b^n}{n^a}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^a \cdot \frac{1}{b} = \frac{1}{b}.$$

So we get 3 cases:

(i) $0 < b < 1$: Diverges by ratio test.

(ii) $1 < b$: Converges " "

(iii) $b = 1$ Diverges because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^a = \infty \neq 0.$$

The value of a doesn't affect convergence.

12. $a_n = \frac{n^a}{n!}$ $\quad a > 0.$ Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^a}{(n+1)!} \cdot \frac{n!}{n^a}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^a \cdot \frac{1}{n+1} = 1 \cdot 0 = 0.$$

So converges for every a.

13. $a_n = \dfrac{a^n}{n!}$ $a > 0$. Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a^{n+1}}{(n+1)!} \cdot \frac{n!}{a^n}$$

$$= \lim_{n\to\infty} \frac{a}{n+1} = 0.$$ So converges for every a.

14. $a_n = \dfrac{n!}{n^n}$ Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{(n+1)\cdot(n+1)^n} \cdot \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

So the series converges. (It is interesting to note that this series behaves roughly like a geometric series with common ratio $\frac{1}{e}$.

15. $a_n = \left(\dfrac{\ln n}{n}\right)$

For $n \geq 1$, $\ln n < n$, so $\dfrac{\ln n}{n}$ 1.

Thus for $n \geq 2$, we have

$$\left(\frac{\ln n}{n}\right)^n < \left(\frac{\ln n}{n}\right)^2.$$

Thus our series here is smaller, term by term, than the one treated in Problem 9, which converged. So the series converges.

16. $a_n = (-1)^n \dfrac{\tan^{-1} n}{n}$

Since $\tan^{-1} n > 0$ for $n > 0$, this series alternates. To apply the alternating series test, we need first to show $\lim\limits_{n\to\infty} a_n = 0$. To see this, note that $0 < \tan^{-1} n < \pi/2$ for all $n > 0$. Therefore

$$0 \leq \lim_{n\to\infty} \frac{\tan^{-1} n}{n} \leq \lim_{n\to\infty} \frac{\pi}{2n} = 0.$$

So $\lim\limits_{n\to\infty} a_n = 0$.

We must also show that

$$|a_{n+1}| < |a_n| \quad \text{for all } n,$$

i.e. that $|a_n|$ is a decreasing function of n for sufficiently large n. To see this, just differentiate with respect to n:

$$D_n(|a_n|) = D_n\left(\frac{\tan^{-1} n}{n}\right) = \frac{\frac{n}{1+n^2} - \tan^{-1} n}{n^2}.$$

Since $\frac{n}{1+n^2}$ approaches 0, while $\tan^{-1} n$ approaches $\pi/2$, we see that the numerator $\frac{n}{1+n^2} - \tan^{-1} n$ eventually becomes (and stays) negative. So for sufficiently large values of n, $|a_n|$ is a decreasing function of n. Thus by the alternating series test, the series converges.

17. $a_n = \dfrac{(2n)!}{n^n}$ Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)^n} \cdot \frac{n^n}{(2n)!}$$

$$= \lim_{n\to\infty} 2(2n+1)\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} 2(2n+1) \cdot \frac{1}{(1+\frac{1}{n})^n} = \infty \cdot \frac{1}{e} = \infty.$$

So it diverges.

18. $a_n = \dfrac{n!}{n^{\frac{1}{2}n}}$ Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{\frac{1}{2}(n+1)}} \cdot \frac{n^{\frac{1}{2}n}}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{(n+1)^{1/2}(n+1)^{\frac{1}{2}n}} \cdot \frac{n^{1/2 n}}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{\sqrt{n+1}} \left(\frac{n^n}{(n+1)^n}\right)^{1/2}$$

$$= \lim_{n\to\infty} \sqrt{n+1} \lim_{n\to\infty} \sqrt{\frac{1}{(1+\frac{1}{n})^n}}$$

$$= \sqrt{\frac{1}{e}} \cdot \lim_{n\to\infty} \sqrt{n+1} = \infty > 1.$$ So diverges.

19. $a_n = \dfrac{n!}{n^{an}}$ $a > 0$. Ratio test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{an+a}} \cdot \frac{n^{an}}{n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)\; n!}{(n+1)^a \;(n+1)^{na}} \cdot \frac{n^{na}}{n!}$$

$$= \lim_{n\to\infty} (n+1)^{1-a} \left(\frac{n^n}{(n+1)^n}\right)^a$$

$$= \lim_{n\to\infty} (n+1)^{1-a} \quad \lim_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})^n}\right)^a$$

$$= \frac{1}{e^a} \lim_{n\to\infty} (n+1)^{1-a}$$

This is $\begin{cases} 0 & \text{if } a > 1 \text{ so converges} \\ \frac{1}{e^a} < 1 & \text{if } a = 1 \text{ so converges} \\ \infty & \text{if } a < 1 \text{ so diverges} \end{cases}$

20. $a_n = (-1)^n \left(\dfrac{2 - \cos n\pi}{n}\right)$

Recall $\cos(n\pi) = (-1)^n$.

Writing out the first few terms of this alternating series, we obtain

$$-3 + \frac{1}{2} - \frac{3}{3} + \frac{1}{4} - \frac{3}{5} + \frac{1}{6} - \dots + \dots$$

20 (cont.)

This series does not satisfy the alternating series test, because $|a_n|$ is not a decreasing function of n. In fact, to see the series diverges, use the following device: look at the series in groups of two terms each, noting that each group of two has the form

$$b_n = -\frac{3}{2n-1} + \frac{1}{2n},$$

where

$$b_1 = a_1 + a_2 = -3 + \frac{1}{2}$$

$$b_2 = a_3 + a_4 = -\frac{3}{3} + \frac{1}{4}$$

$$b_3 = a_5 + a_6 =, \quad \text{etc.}$$

Now if $\sum a_n$ converges, $\sum b_n$ should also converge. However

$$b_n = -\frac{3}{2n-1} + \frac{1}{2n} = \frac{-6n + 2n-1}{(2n-1)(2n)}$$

$$= \frac{-4n-1}{2n(2n-1)}.$$

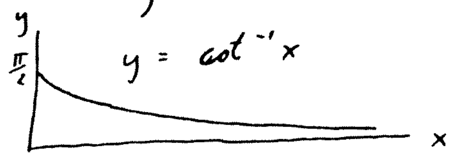
We can see $\sum b_n$ diverges by comparing it to the divergent series $\sum \frac{1}{n}$.

(cont.)

20 (cont.)

The reader should verify this, using the ratio comparison test. Finally, since $\sum b_n$ diverges, $\sum a_n$ does as well. This problem demonstrates that the usual tests do not always suffice to demonstrate convergence or divergence.

21 $a_n = \cot^{-1} n$ We use the integral test because $\cot^{-1} n$ is a positive, decreasing function:



$$\int_0^\infty \underbrace{\cot^{-1} x}_{u} \; \underbrace{dx}_{dv}$$

Integration by parts
$u = \cot^{-1} x$
$du = -\frac{1}{x^2+1}\,dx$
$dv = dx$
$v = x$

$$= uv - \int v\,du =$$

$$x\cot^{-1} x + \int \frac{x}{1+x^2}\,dx = x\cot^{-1} x + \frac{1}{2}\ln(1+x^2)\Big|_0^\infty$$

$$= \lim_{N\to\infty} \left(N\cot^{-1} N + \frac{1}{2}\ln(1+N^2) \right)$$ cont.

21 (cont)

Taking these limits separately:

$$\lim_{N\to\infty} N\cot^{-1} N = \lim_{N\to\infty} \frac{\cot^{-1} N}{1/N}$$

$$= \lim_{N\to\infty} \frac{-\frac{1}{N^2+1}}{-\frac{1}{N^2}} = \lim_{N\to\infty} \frac{N^2}{N^2+1} = 1.$$

$$\lim_{N\to\infty} \frac{1}{2}\ln(1+N^2) = \infty.$$ So the integral and, thus, the series diverges.

22. $a_n = \dfrac{(n!)^2\, 2^n}{(2n)!}$ Ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2\, 2^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2 \cdot 2^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 (n!)^2 \cdot 2 \quad (2n)!}{(2n+2)(2n+1)(2n)!\ (n!)^2}$$

$$= \lim_{n\to\infty} \frac{2(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{2}.$$ So the series converges.