Mathematics Support Capsules

GRAPHING WITH CALCULUS

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Prerequisites: how to calculate limits and derivatives .

Calculus is a tool that provides additional information for graphing information that cannot be obtained by using the non-calculus techniques reviewed below. When used together, the calculus (5-7) and non-calculus (1-4,8) techniques allow us to graph almost any function given.

Non-calculus techniques: — the following is a brief review of graphing techniques that do not require calculus. The accompanying explanations are very brief. More detailed explanations and exercises are given in the GRAPHING Capsules. If you have any trouble understanding these explanations, or for a good review of graphing, I strongly suggest looking into GRAPHING.

Consider a function $y = f(x)$

1) x and y intercepts — these are the points where the curve crosses the x and y axes respectively. (i.e., x-intercept is where $y=0$ and y-intercept is where $x=0$).

To find: x-intercept, set $f(x) = 0$, solve for x.

: y-intercept, set $x = 0$, solve for $y = f(0)$.

2) Symmetry — this is when one part of the graph is an exact mirror image of the other part.

There are two types:

a) Symmetry about the y-axis. This occurs when $f(a) = f(-a)$, where a is any number.

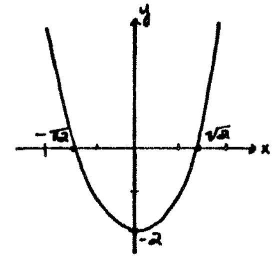
e.g. $f(x) = x^2 - 2$

$f(3) = 9 - 2 = 7$

$f(-3) = 9 - 2 = 7$

$f(3) = f(-3)$

symmetry about y-axis



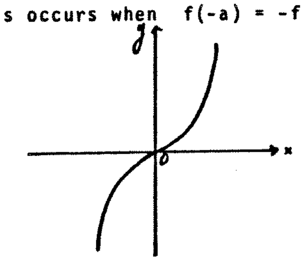
b) Symmetry about the origin - this occurs when $f(-a) = -f(a)$ where a is any number.

e.g. $f(x) = x^3$

$f(3) = 27$

$f(-3) = -27$

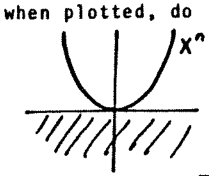
symmetry about the origin



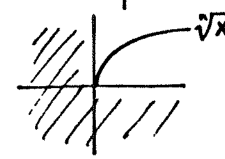
Symmetry is very important in that if we know symmetry exists, we need only to find half the graph and we will also find the other half.

3) Inadmissible regions — there are functions that, when plotted, do not make use of the entire x-y plane

e.g. $f(x) = x^n$, where n is any even number will limit $f(x)$ to only positive values



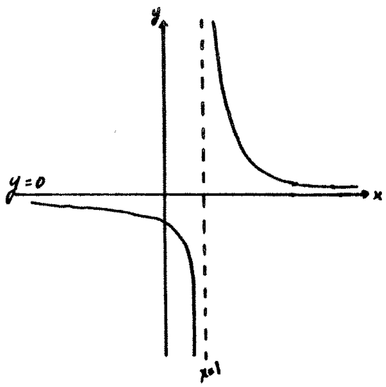
$f(x) = x^{\frac{1}{n}}$ or $\sqrt[n]{x}$, where n is any even number will limit $f(x)$ to only positive values and x to only positive values.



If we can determine that there is an inadmissible region on our graph, we can ignore this region.

4) <u>Asymptotes</u> — these are lines that a function approaches but never touches.

e.g. $f(x) = \frac{1}{x-1}$. This function is not defined at $x = 1$. x will get close to 1 but not hit it. So $x = 1$ is an asymptote



If the numerator of $f(x)$ is always 1, then y will never be 0, although it gets closer to 0 as x approaches extreme values. So $y = 0$ is a second asymptote.

Asymptotes are important because they are a part of the picture. They act as a guide as to where the graph should be.

<u>Calculus Techniques:</u> — The two topics of Calculus most often found in graphing are <u>limits</u> and <u>derivatives</u>.

<u>note</u>: A more in depth discussion and review with exercises for limits may be found in Ray Bacon's LSC Mathematics Learning Module II, <u>Limits: An Intuitive Approach</u>.

<u>Limits</u>: Limits tell us what is going on as x gets closer to a number. In the function above, $f(x) = \frac{1}{x-1}$, we can use limits to determine the behavior of the curve as it gets closer to $x = 1$.

$\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$	$\lim_{x \to 1^-} \frac{1}{x-1} - \infty$
as $x \to 1$ from the positive side, $f(x)$ gets very large positively	as $x \to 1$ from the negative side, $f(x)$ gets very large negatively

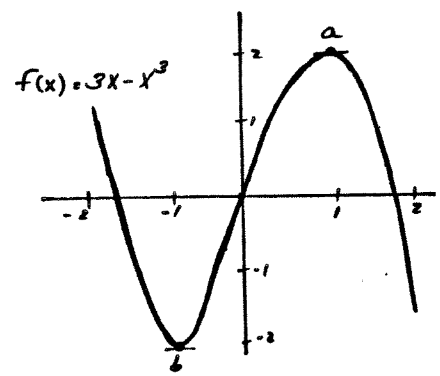
5) <u>$x \to \infty$ and $x \to -\infty$</u>.

Limits also tell us what happens at the extremes.

As x gets larger positively	As x goes to the negative extreme,
$\lim_{x \to \infty} \frac{1}{x-1} = 0$	$\lim_{x \to -\infty} \frac{1}{x-1} = 0$
$f(x)$ gets closer to 0.	$f(x)$ gets closer to 0.

<u>Derivatives</u>: — By far the most useful topics in Calculus for graphing are the derivatives.

6) <u>First derivative [f'(x)]</u> - the first derivative gives <u>slope</u>.



In the given function $f(x) = 3x - x^3$, the two points (a,b) where the curve stops going in one direction and turns toward the other are the the maximum and minimum points.

At point a, the curve stops increasing (positive slope) and starts decreasing (negative slope). This is a relative maximum point.

At point b, the curve stops decreasing and starts decreasing. This is a relative minimum point.

These max. and min. points have a slope = 0. (because horizontal lines have slope = $\frac{\Delta y}{\Delta x}$ where $\Delta y = 0$, so that $\frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$). To find the max and min points, take the first derivative of the function and set it equal to 0. Then solve for the x-values.

for our example.

$f(x) = 3x - x^3$

$f'(x) = 3 - 3x^2 = 0$

$3 - 3x^2 = 0$

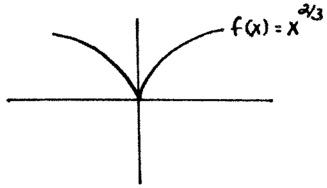
$3 = 3x^2$

$1 = x^2$

$\pm 1 = x$

→ $f(1) = 2$, $f(-1) = -2$

our max and min points are (1,2) and (-1,-2).



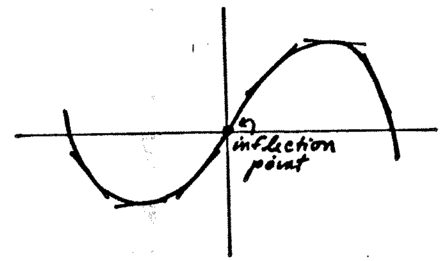
Another possible place for relative max and min points is where a derivative does not exist. (e.g., $f(x) = x^{2/3}$ has no derivative but a relative min. at x=0). So, check points where $f'(x)$ does not exist as well. Our example $f(x) = 3x - x^3$ has no such points.

The first derivative tells us the location of the max and min points. But it does not tell us which one is a max and which one a min point. This is the job of the second derivative.

7) <u>Second derivative</u> — [$f''(x)$] = The second derivative is the derivative of the first derivative.

The second derivative shows <u>concavity</u>. Given:

$f(x) = 3x - x^3$

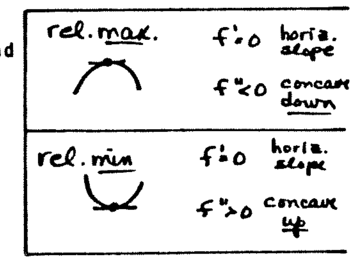


The curve is concave up (opens upward) where $f'(x)$ (slope) is always increasing from left to right and <u>$f''(x) > 0$</u>

The curve is concave down (opens downward) where $f'(x)$ (slope) is always decreasing from left to right and <u>$f''(x) < 0$.</u>

A point where the concavity changes is called an <u>inflection point</u>. At this point, <u>$f''(x) = 0$.</u>

A point with its <u>first</u> derivative equal to 0 and <u>second</u> derivative less than 0 (negative) is a relative maximum. A point with its <u>first</u> derivative equal to 0 and <u>second</u> derivative greater than 0 (positive) is a relative minimum.



in our example:

$f'(x) = 3 - 3x^2$

$f''(x) = -6x.$

<u>our critical points x = +1, x = -1</u>

$f''(1) = -6$ rel. max at x=1

$f''(-1) = -6(-1) = 6$ rel. min at x=-1

$f''(x) = -6x = 0$

$-6x = 0$

$x = 0$ ← inflection point

The following table summarizes the relationship between the first and second derivative.

First Derivative (f'(x))	Second Derivative (f"(x))	x is a ...	Diagram
0	-	rel.max.pt.	
0	+	rel.min.pt.	
undefined	undefined	may be a rel. max. or min.pt.	e.g.,  f(x)= $x^{2/3}$ or  f(x)= |x|
+	0	infl.pt.	 or 
-	0	infl.pt.	 or 
0	0	...mystery. It has a horizonal slope. But you need more info. to determine shape.	

Dotted lines (------) indicate slope at the point.

8) Point plotting — This is where you find f(x) by plugging in x. This is the method of graphing to use when you are given a function with a point or a set of points where all hell breaks loose and nothing else works.

Before we start graphing, let's list again all the things we need to know.

1) x and y intercepts
2) symmetry
3) inadmissable regions
4) asymptotes
5) $x \to \infty$, $x \to -\infty$
6) first derivative information (slopes, max-min points)
7) second derivative information (concavity, inflection points)
8) other points that will help graph the function

Note: There will be times when either some of these things don't exist or they are too hard to find. Don't spend too much time on finding them. Only look for them at the end when you have more time and/or know they exist. This note does not include (6). Always look for max-min points.

This list may seem long in the beginning, but with practice it gets much shorter and easier.

A final note: Graphing problems are overdetermined. The above information (points, slopes, concavity, etc.) should fit together. If it doesn't, check for mistakes.

Example:

Graph $f(x) = x^3 - 3/2x^2 - 6x + 5$

pts. to consider

1) x-y intercepts $y = 0 \Rightarrow x = ?$ hard to tell

$x = 0 \Rightarrow y = 5$ (0,5)

2) symmetry - try f(1) and f(-1)

$f(1) = (1)^3 - 3/2(1)^2 - 6(1) + 5 = -3/2$ (1, -3/2)

$f(-1) = (-1)^3 - 3/2(-1)^2 - 6(-1) + 5 = 8\ 1/2$ (-1, 8 1/2)

no symmetry

3) inadmissable regions - probably not.

4) asymptotes - none.

5) $x \to \infty$, $x \to -\infty$

$$\lim_{x \to \infty} \underbrace{x^3}_{+} - 3/2x^2 - 6x + 5 = +\infty$$

$$\lim_{x \to -\infty} \underbrace{x^3}_{-} - 3/2x^2 - 6x+5 = -\infty$$

6) max-min points

$$f'(x) = 3x^2 - 3x - 6 = 0$$
$$3(x^2-x-2) = 0$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x\ 2) = 0$$
$$x = -1, +2$$

(-1, 8 1/2)

(2, -5)

$$f''(x) = 6x - 3 = 0$$
$$6x = 3$$
$$x = 1/2$$

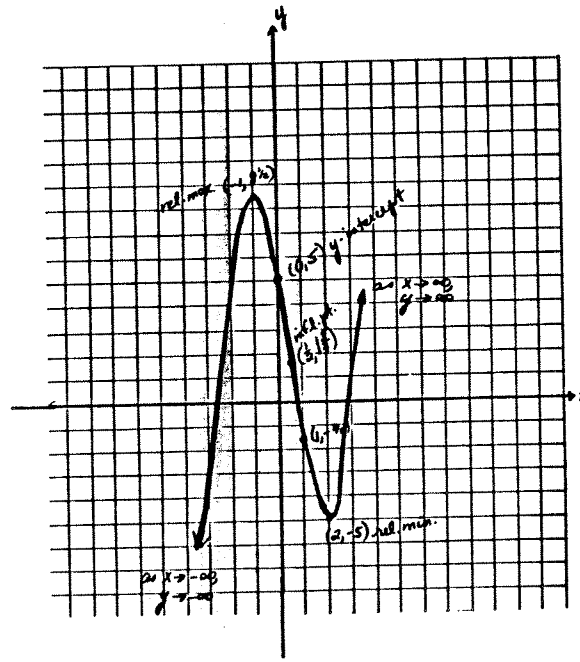
(1/2), 1 3/4)

$f''(-1) = 6(-1) - 3 = -9$ (x = -1 is a rel. max)

$f''(2) = 6(2) - 3 = 9$ (x=2 is a rel. min.)

x	f(x)	f'(x)	f"(x)	
0	5	-	-	} for these points. f'(x) and f"(x) are found as a check.
1	-3/2	-	+	
-1	8 1/2	0	- (max)	
2	-5	0	+ (min)	
1/2	1 3/4	-	0 infl.pt.	} or ⌇

You don't have to find the exact number, just the sign.



Another Example: $f(x) = (x-1)(x)^{2/3}$

1) $x = 0,\ y = 0$ (0,0)

$y = 0,\ x = 0,1$ (1,0)

2) symmetry - none

3) inadmissable regions-hard to tell

4) asymptotes - none

5) $x \to \infty,\ y \to \infty$

$x \to -\infty,\ y \to -\infty$

6) $f'(x) = (x-1)2/3(x)^{-1/3} + (x)^{2/3}(1)$

$$= \frac{2(x-1)}{3(x)^{1/3}} + x^{2/3} = 0$$

(0,0)-(f'(x) not defined at (0,0) - watch out!

$$\frac{2(x-1)}{3(x)^{1/3}} = -x^{2/3}$$

$2x-2 = -3x$

$-2 = -5x$

$2/5 = x$

(2/5, .326) need calculator to find f(2/5)

7) $f''(x) = (x-1)(-2/9(x)^{-4/3}) + 2/3x^{-1/3}(1) + 2/3x^{-1/3}$

$$= \frac{-2(x-1)}{9x^{4/3}} + 2\left(\frac{2}{3x^{1/3}}\right)$$

f" (0) - undefined, f"(2/5) is + (rel. min.)

$$f''(x) = \frac{-2(x-1)}{9x^{4/3}} + \frac{4}{3(x)^{1/3}} = 0$$

$$2x-2 = \frac{4(9x^{4/3})}{3x^{1/3}}$$

$2x-2 = 4(3x)$

$2x-2 = 12x$

$-2 = 10x$

$-1/5 = x$

(-1/5, .274) - infl. pt.
Calculator

x	f(x)	f'(x)	f"(x)	
0	0	undef.	undef.	
1	0	+	+	
2/5	-.326	0	+	
-1/5	-.274	+	0	ʃ∩ᐟ
-1			−	
-1/10			+	ᐟ

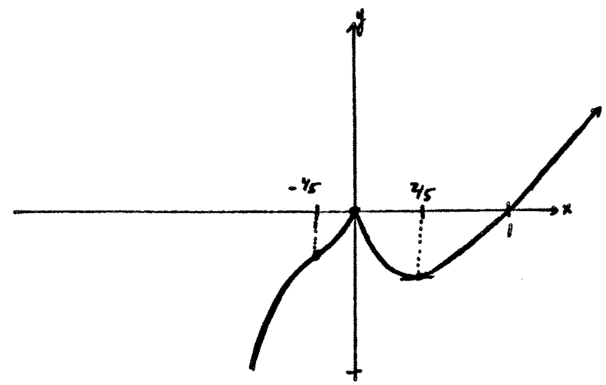
What happens at (0,0)?
We can find out by seeing what happens around it.

a) There is a rel. min. pt. on the positive side (x = 2/5). Since there are no max points on this side, the curve must come <u>up</u> to x=0. (We know that f(x) exists)

b) There are no max or min point on the left side. But we know an inflection point exists at (-1/5, -/274) with a positive slope. The curve must also rise from (y=0) to (y=0) from the left side.

The last thing to find is the shape of the curve on the left side. Which of the above curves at an inflection points is right? You can find out by find the concavity of points to the left and right of the infl. pt. Try f"(-1) and f"(-1/10).

To put the graph together.



Exercise

Graph the following:

1) $f(x) = 2x^2 + x - 1$

2) $f(x) = \frac{x^3}{3} - 2x^2 - 5x$

3) $f(x) = x^6$ and $f(x) = x^5$ on separate graphs and compare $f'(0)$, $f''(0)$, and the pictures at these points.

4) $f(x) = 2\sqrt{x} - x$

5) $f(x) = x^5 - 5x^3$

6) $f(x) = (x^2+1)^5$

7) $f(x) = \frac{x^2}{x-1}$

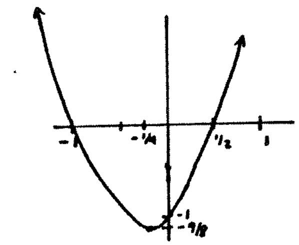
8) $f(x) = \sqrt{x^3+1}$

Answers on next page.

The following are only partial answers to the problems. Only the more difficult and necessary calculations are worked out. The remaining information can be found in the graphs.

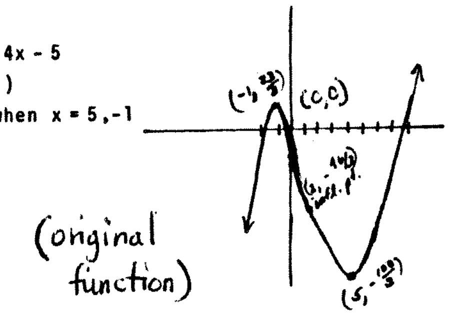
1) $f'(x) = 4x+1$
$f''(x) = 4$

$f(x) = 2x^2 + x - 1$
$= (2x-1)(x+1)$
$x = 1/2, -1, y = 0$

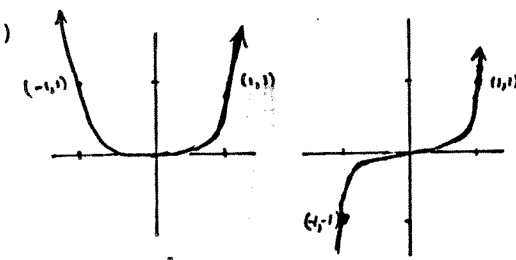


2) $f'(x) = x^2 - 4x - 5$
$f''(x) = 2x - 4$

$f'(x) = x^2 - 4x - 5$
$= (x-5)(x+1)$
$f'(x) = 0$ when $x = 5, -1$



3)

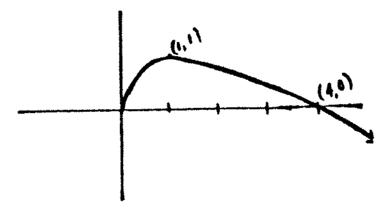


$f(x) = x^6$	$f(x) = x^5$
$f'(0) = 0$	$f'(0) = 0$
$f''(0) = 0$	$f''(0) = 0$

The slope at x=0 is horizontal for both graphs. You need more info. to identify the behavior of the graph.

4) $f'(x) = \frac{1}{\sqrt{x}} - 1$
$f'(x) = -1/2(x)^{-3/2}$

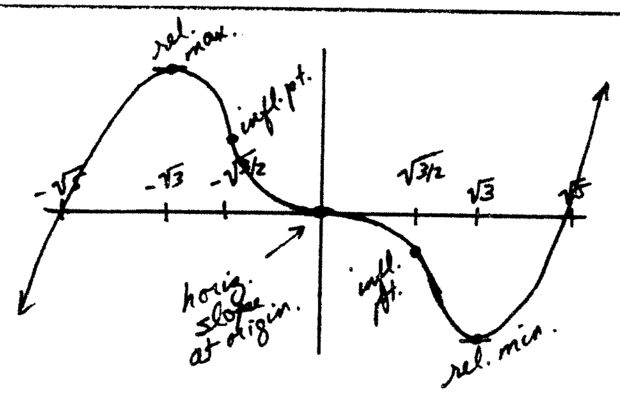
x=0 is an endpoint because: inadmissable region is $x < 0$



5) $f'(x) = 5x^4 - 15x^2$

$f''(x) = 20x^3 - 30x$

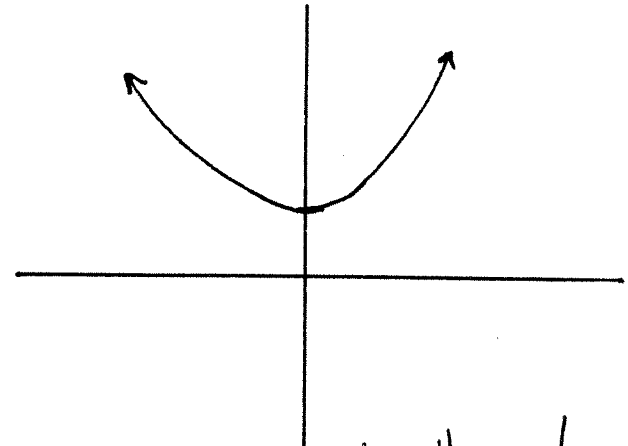
There is symmetry about origin; therefore x=0 is an infl. pt.



6) $f'(x) = 5(x^2+1)^4(2x)$ = 0 at origin

$f''(x) = 80x^2(x^2+1)^3 + 10(x^2+1)^4$ (always +)

Inadmissable region : $y < 1$
symmetry about y-axis



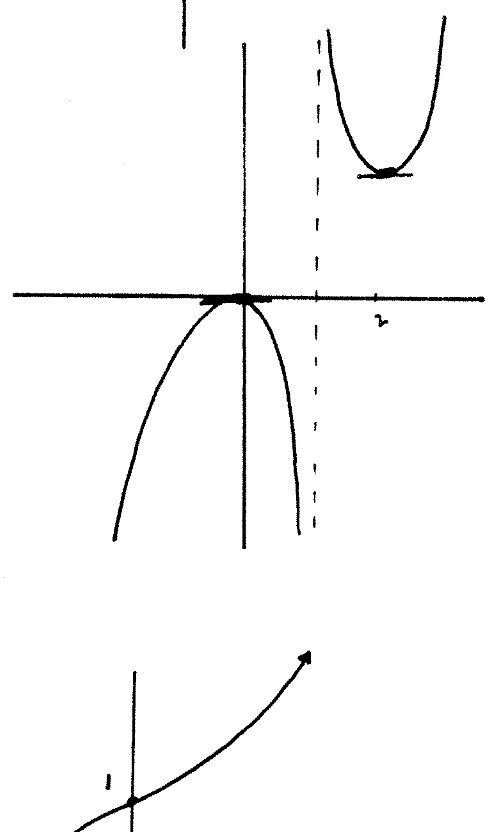
7) $f'(x) = \frac{x^2-2x}{(x-1)^2}$ (simplified) = 0 at $x=0$, $x=2$

$f''(x) = \frac{2x-2}{(x-1)^2} - \frac{2x^2-4x}{(x-1)^3}$ (simplified)

asymptote at x=1

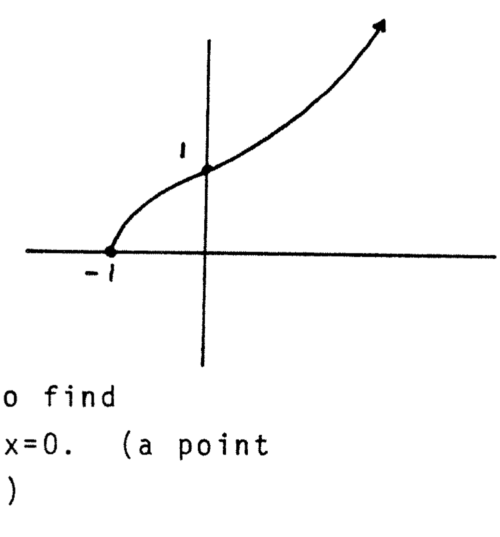
$x \to 1^+ \Rightarrow f(x) \to +\infty$

$x \to 1^- \Rightarrow f(x) \to -\infty$



8) $f'(x) = \frac{3x^2}{2\sqrt{x^3+1}}$

$f''(x) = \frac{3x}{(x^3+1)^{1/2}} - \frac{9x^4}{4(x^3+1)^{1/2}}$



To find the shape of the curve, you need to find concavity of points greater and less than x=0. (a point to the right and one to the left is enough)