## EXPONENTS AND INTEGERS AS EXPONENTS

A product is the result of a multiplication of two or more terms called factors (i.e., 15 = 3 x 5 or 60 = 3 x 4 x 5). When a product is composed of several factors which are the same (i.e., 9 = 3 x 3 or 64 = 4 x 4 x 4), there is a shorter method of expressing that product. Instead of writing the one factor over and over with multiplication signs (i.e., 4 x 4 x 4), simply write the repeated factor once and above it, to the right, write the number of times it is multiplied (i.e., 4 x 4 x 4 = $4^3$). This shorter expression is called a power. In the expression, the factor is called the base and the number of multiplications is called the exponent.

In general then...

if n and b are positive numbers, then in the expression $b^n$

$$b^n = \underbrace{b \times b \times b \times b \times \ldots \times b}_{n \text{ factors of } b}$$

n is the exponent

b is the base

$b^n$ is the $n^{th}$ power of b

or, b raised to the $n^{th}$

Numbers written as powers may be used directly in computation. There are some rules which may be followed when multiplying and dividing powers and, as with all rules of computation, these may be shown to be true from the definitions of the terms involved. Following are the rules for multiplication and division of powers and their verification from the definition of powers given above.

I. When powers have the same base (involve the same factor), their product and quotient (the result of their division) will also have that same base. In multiplication, the exponent in the product will be the sum of the exponents in each of the powers while in division, the exponent in the quotient will be the difference of the exponent of the divisor power (power divided by) from the exponent of the dividend power (power divided into).

Example 1 - Multiplication

$2^3 \times 2^5$ (base: 2, product exponent; sum of 3 and 5)

$$2^3 \times 2^5 = 2^{3+5} = 2^8$$

in general... $b^n \times b^k = b^{n+k}$



Example 2 - Division

$4^7/4^3$ (base: 4, quotient exponent: difference of 7 minus 3)

$$4^7/4^3 = 4^{7-3} = 4^4$$

in general... $b^n/b^k = b^{n-k}$

It can easily be seen how these rules follow from the definition of a power...

1) $2^3$ means 2 x 2 x 2 and $2^5$ means 2 x 2 x 2 x 2 x 2 so that $2^3 \times 2^5$ means (2 x 2 x 2) x (2 x 2 x 2 x 2 x 2).
Following the rules for multiplication, the above product equals 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2, which can be written as the power, $2^8$.

2) $4^7$ means 4 x 4 x 4 x 4 x 4 x 4 x 4 and $4^3$ means 4 x 4 x 4 so that $4^7/4^3$ means $\dfrac{4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4}$
Following the rules for canceling in division, the above quotient equals 4 x 4 x 4 x 4, which written as a power is $4^4$.

These rules are shown only in specific examples; however, the same line of reasoning may be used to show that these rules apply in all cases.

II. When powers have the same exponent, their product and quotient will have that same exponent. In multiplication, the base in the product will be the product of the bases of the powers, while in division, the base of the quotient will be the quotient of the bases of the powers.

Example 1 - Multiplication

$2^3 \times 3^3$ (exponent: 3, product base: product of 2 and 3)

$$2^3 \times 3^3 = (2 \times 3)^3 = 6^3$$

in general... $b^n \times d^n = (b \times d)^n$

Example 2 - Division

$50^4/5^4$ (exponent: 4, quotient base: quotient of 50 and 5)

$50^4/5^4 = (50/5)^4 = 10^4$

in general... $b^n/d^n = (b/d)^n$

Again, these rules follow from the definition of a power.

1) $2^3$ means $2 \times 2 \times 2$ and $3^3$ means $3 \times 3 \times 3$ so that $2^3 \times 3^3$ means $(2 \times 2 \times 2) \times (3 \times 3 \times 3)$.

Following the rules of multiplication, this may be rewritten $2 \times 2 \times 2 \times 3 \times 3 \times 3$ and rearranged to $2 \times 3 \times 2 \times 3 \times 2 \times 3$ (in multiplication the order of multiplying is not important). This last product may be written $(2 \times 3) \times (2 \times 3) \times (2 \times 3)$ which written as a power is $(2 \times 3)^3$ or $6^3$.

2) $50^4$ means $50 \times 50 \times 50 \times 50$ and $5^4$ means $5 \times 5 \times 5 \times 5$ so that $50^4 \div 5^4$ means $\frac{50 \times 50 \times 50 \times 50}{5 \times 5 \times 5 \times 5}$. Following the rules for multiplication within a division, this may be rewritten $\frac{50}{5} \times \frac{50}{5} \times \frac{50}{5} \times \frac{50}{5}$ which written as a power is $(50/5)^4$ or $10^4$.

III. When a power is raised to an exponent, the base in the resulting power is the same as the base of the base power. The exponent in the resulting power is the product of the exponent of the base power and the exponent to which that power is raised.

Example: Power to a power

$(7^4)^3$ (base of base power: 7, resulting exponent: product of 4 and 3)

$(7^4)^3 = 7^{4 \times 3} = 7^{12}$

in general... $(b^n)^k = b^{nk}$

From the definition of a power...

$7^4$ means $7 \times 7 \times 7 \times 7$ and $(7^4)^3$ means $(7 \times 7 \times 7 \times 7) \times (7 \times 7 \times 7 \times 7) \times (7 \times 7 \times 7 \times 7)$ following the rules for multiplication, this equals $7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$ which can be written as a power $7^{12}$.



NEGATIVE EXPONENTS

A) When you write the division $2^3/2^5$, what this means from the definition of powers is $\frac{2 \times 2 \times 2}{2 \times 2 \times 2 \times 2 \times 2}$.

Using the rules of division, it is possible to cancel out some of the two's (i.e., $\frac{\not{2} \times \not{2} \times \not{2}}{\not{2} \times \not{2} \times \not{2} \times 2 \times 2}$).

After canceling this expression, it now equals $1/(2 \times 2)$ which could be written $1/2^2$.

B) Following the rules for division with like bases $2^3/2^5$ would give the solution $2^{3-5}$ or $2^{-2}$.

Since both of the above methods of solving the divisions give proper solutions for the division $2^3/2^5$, their answers must be equal (i.e., $1/2^2 = 2^{-2}$). From this we get a definition for powers which have negative exponents...

A base raised to a negative exponent is equal to one over that same base raised to the positive form of that exponent.

in general... $b^{-n} = \frac{1}{\underbrace{b \times b \times b \ldots b}_{n \text{ factors of } b}}$

*NOTE*

Powers of negative exponents follow the same rules of computation given above.

Powers with negative exponents <u>are not negative</u> numbers...They are fractional expressions with powers as numerators.

Zero as an Exponent

It can be seen from the division rules for powers with like bases that when two powers have like bases and like exponents, their quotient will give us the base raised to a zero exponent (i.e., $5^3/5^3 = 5^{3-3} = 5^0$).

We know already that powers with like bases and exponents are equal ($5^3 = 5^3$) and also that when something is divided by its equal, the result is always one ($5^3/5^3 = 1$).

Since both of the above show correct solutions of the division $5^3/5^3$, their answers must be equal ($5^0 = 1$). Further, it will be found that in all cases where a base is raised to a zero exponent, the power equals one.

in general... $b^0 = 1$

## In Summary

The above has been an introduction to the meaning and use of integers as exponents. Since integers are all positive or negative whole numbers and zero, you should be able to use any power with positive or negative whole number or zero exponents in computation. The following are intended for practice in this.

*NOTE* In the problems or in the explanations above, no limitations are put on the types of bases that may be used (i.e., letters, numbers, or symbols) or the number of combinations of operations that may be performed. Examples of some possible problems are given below.

Ex. 1) $x^7 \cdot x^3 \cdot x^5 = x^{7+3+5} = x^{15}$

Ex. 2) $\frac{x^{5n}}{x^n} = x^{5n-n} = x^{4n}$

Ex. 3) $(ab)\ (a^2b^3) = aba^2b^3 = aa^2bb^3 = (a \cdot a^2)\ (b \cdot b^3) =$

$(a^{1+2})\ (b^{1+3}) = a^3b^4$

Ex. 4) $(x^2/y)^3 = (x^2)^3 / (y)^3 = \frac{x^6}{y^3}$

Ex. 5) $\frac{4x^2y^3}{8x^3y} = \frac{4}{8} \cdot \frac{x^2}{x^3} \cdot \frac{y^3}{y} = \frac{1}{2} \cdot x^{2-3} \cdot y^{3-1} = (1/2)x^{-1}y^2$ or

$1/2 \cdot 1/x \cdot y^2/1 = y^2/2x$



## The Significance of Ten as a Common Base (Scientific Notation)

As any number, 10 may be raised to integral exponents and used in computation. The number ten, when used as a base for a power, has special significance since our number system is a decimal number system (based on the powers of ten).

For example:

Take any many digit number like 732.54. The position of each digit signifies a different power of ten as a multiple of the digit. Hundreds, tens, ones, tenths and hundreths places are names based on the powers of $10^2$, $10^1$, $10^0$, $10^{-1}$ and $10^{-2}$. The number 732.54 may then be rewritten

$$7 \times 10^2 + 3 \times 10^1 + 2 \times 10^0 + 5 \times 10^{-1} + 4 \times 10^{-2}$$

using powers of ten and the digit location in the original numbers.

This process of writing out a number using the powers of 10 seems cumbersome in the case of the number 732.54, but let's take another many digit number like 7,000,000. Using the power of ten form to write this number we get $7 \times 10^6$, which is certainly shorter than writing seven million out with all the zeroes.

Sometimes writing a number out in power of ten form can be used in a modified manner. For instance, the number 73,000,000 could be written as $7 \times 10^7 + 3 \pm 10^6$, but it could also be written in the simpler forms $7.3 \times 10^7$ or $73 \times 10^6$. In this case a modified version of writing the number in power of ten form is the simplest.

In order to aid in using the power of ten form of writing a number, the power of ten associated with a digit location in a number must be known. This is shown below for the example number 852,347.619587

| Power of 10 associated with the place of digit | etc. | 5 | 4 | 3 | 2 | 1 | 0 | -1 | -2 | -3 | -4 | -5 | -6 | etc. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | ↓ | |
| Example Number | | 8 | 5 | 2, | 3 | 4 | 7. | 6 | 1 | 9 | 5 | 8 | 7 | |

Following this line of reasoning, the number .063 can be written as $63 \times 10^{-3}$ or $6.3 \times 10^{-2}$ depending on which place digit you intend to use as a reference.

## Standard Number Form (SNF or Scientific Notation)

In science, economics or many places where large quantities of numbers of varying magnitudes are often encountered, the power of ten form of expressing numbers is preferred for speed, accuracy and form. The convention (agreed form) for these power of 10 numbers is that they be written with their number part (non power of ten part) possessing one digit to the left of the decimal point. (i.e., 7300 = $7.3 \times 10^4$ and .0051 = $5.1 \times 10^{-3}$).

## Working With SNF

Powers of ten are calculated as any powers with like bases (i.e., $10^3 \times 10^5 = 10^8$ or $10^3/10^5 = 10^{-2}$). Standard Number Form numbers contain a number part (with one digit to the left of the decimal) and a power of ten part. When computing with these numbers, some basic rules must be followed to avoid some common mistakes.

### I. Multiplying and Dividing

Since a SNF number is simply a product, the product or quotient of SNF numbers may be separated following the rules of multiplication and division. The separation should occur in such a way as to segregate all powers of ten from all number parts.

Example 1 $(3.2 \times 10^4) \times (4.5 \times 10^{-3}) = 3.2 \times 10^4 \times 4.5 \times 10^{-3} =$

$3.2 \times 4.5 \times 10^4 \times 10^{-3} = (3.2 \times 4.5) \times (10^4 \times 10^{-3})$

Using rules for multiplying powers, this equals $(3.2 \times 4.5) \times 10^1$.

Example 2 $(3.2 \times 10^4) \div 4.5 \times 10^{-3} = \frac{3.2 \times 10^4}{4.5 \times 10^{-3}} = \frac{3.2}{4.5} \times \frac{10^4}{10^{-3}}$

Using rules for dividing powers, this equals $\frac{3.2}{4.5} \times 10^7$.

### II. Addition and Subtraction

Since powers are only special types of products, there is no shorthand way to add or subtract them. For instance, $10^2 + 10^3$ is the same as 100 + 1000 which equals 1100 (or $1.1 \times 10^3$, or $11 \times 10^2$). There is no rule to follow for addition and subtraction as there is for multiplication and division.



To make addition and subtraction of SNF numbers easier (not writing all out as standard numbers), the numbers may be converted to have the same power of ten. Once this is done, the number part may be added or subtracted and the result will have the same power of ten as the numbers which were added or subtracted.

Example 1) $5.31 \times 10^3 + 6.42 \times 10^4$

convert to: $5.31 \times 10^3 + 64.2 \times 10^3$

add number parts: $5.31 + 64.2 = 69.51$

result: $69.51 \times 10^3$

Example 2) $6.24 \times 10^{-2} - 5.43 \times 10^{-4}$

convert to: $6.24 \times 10^{-2} - .0543 \times 10^{-2}$

subtract number parts: $6.24 - .0543 = 6.1857$

result: $6.1857 \times 10^{-2}$

Practice may be needed in converting numbers into scientific notation or changing a number from one power of 10 into another power of 10.

Below are some rules to follow:

To convert a number into SNF, first identify the place occupied by the digit which will be to the left of the decimal point in the number. The number of this place from the list on page 6 (power of 10), will be the power of ten of this SNF number.

Example 1) 52000 the number will be 5.2
The 5 occupies the fourth place in power of ten
SNF is $5.2 \times 10^4$...

Example 2) .000057 the number will be 5.7
50 occupies the -5 place in power of ten
SNF is $5.7 \times 10^{-5}$

When converting a number from one power of ten to another, calculate the difference between the exponents on the two powers. If you are converting from a larger power to a smaller power of ten, simply shift the decimal point to the right (making the number larger) the number of places represented by the difference between the exponents. If you are converting from a smaller power to a larger power of ten, shift the decimal point to the left (making the number smaller) the number of places represented by the difference between the exponents.

Example 1) $5.23 \times 10^4$ convert to $10^2$ number

Difference: $4-2 = 2$, larger to smaller the number is $523. \times 10^2$

Example 2) $78.41 \times 10^{-7}$ convert to $10^{-5}$ number

Difference $(-5) - (-7) = 2$, smaller to larger the number is

$.7841 \times 10^{-5}$

Problem Set I - Exponents and Powers of 10

1. Write the following numbers in standard number form, (S.N.F.)

   (a) 420,000,000 (b) 20,006 (c) .0072

   (d) $.51 \times 10^5$ (e) 3.6 million (f) 325 million billion

2. It has been estimated that the average abundance of radium in the earth's crust is about one part per million million. Write one million million in standard number form.

3. Change to non-exponential values:

   (a) $1.86 \times 10^5$ (b) $.003 \times 10^4$ (c) $642 \times 10^{-3}$

   (d) $.003 \times 10^{-3}$ (e) $5.12 \times 10^0$ (f) $(4.24)^0$

4. The diameter of a hydrogen molecule is $5.8 \times 10^{-8}$ centimeters. If we placed 100 million of them in a row, just touching, how many millimeters long would the row be?

   10 mm. = 1 cm., or
   1 mm. = 1/10 cm.

5. We know that $10^{-1}$ is equal to 0.1

   Show that $4^{-1}$ is not equal to 0.4

6. Evaluate: $3^{-1} \times 3^2 + 3^0 + 9^8 \times 9^{-7}$

7. Evaluate: $\frac{(40)^3(4)^3}{(80)^3}$

8. Simplify, expressing your answer with positive exponents. (a) $(a^0b^2c^5)^3$ (b) $\frac{c^4b^2c^5}{a^4b^5c^{-3}}$

9. Evaluate: $3^2 \times 2^5$ Is there any special rule for this situation?

10. Give answers in S.N.F.

    (a) $(1.4 \times 10^{-5})(2.8 \times 10^{-8})$ (b) $(8.4 \times 10^{-8}) + (1.2 \times 10^{-4})$

    (c) $\frac{9.0 \times 10^{40}}{2.0 \times 10^{43}}$ (d) $\frac{10^7 \times 10^{-8} \times 10^4}{10^{-4} \times 10}$

11. If the mass of an electron is about $9 \times 10^{-28}$ grams, and the mass of a proton is about $1.62 \times 10^{-24}$ grams, about how many times the mass of an electron is the mass of a proton?

12. Using powers of 10, find the value of $(.004)^3$, and write your answer in standard number form.

13. In this problem, the sequence of digits in the answer is given, but the decimal point has not been placed. Using powers of 10, and the rules of exponents, place the decimal point, writing your answer in S.N.F.

(a) $\frac{30,000,000 \times .000012}{9000,000 \times .0002} = 2 \times 10^?$

(b) $\frac{160,000 \times .0002 \times 120}{.006 \times 20,000 \times .00032} = 1 \times 10^?$

(c) $(3..1 \times 10^{-5}) \cdot (1.0 \times 10^{-5})^2$
$= 3.1 \times 10^?$

14. Add: (a) $5.62 \times 10^{-5} + 3.17 \times 10^{-5}$

(b) $6.3 \times 10^{-3} + 520 \times 10^{-5} + .0028$

15. Which of the following is the largest number?

(a) $(1000)^{10}$ (b) One billion, raised to the third power (c) $(100)^{10}$ (d) $10^{32}$

Suggestion: write each as a power of 10, and compare.

16. Given the conversion units: 1 micron = $10^{-4}$ cm., 1 millimicron = $10^{-7}$ cm., and 1 angstrom = $10^{-8}$ cm.,

(a) How many millimicrons are there in 1 micron?
(b) How many angstroms are there in 1 micron?

17. Write $e^{a+bi}$ as the product of two literal numbers (letters).
Hint: $e^? \times e^?$ will give the above results?

18. $(.02 \times 10^{-4})^3 (2 \times 10^3)^2 = 1 \times 10^?$

19. The diameter of a particle is $14 \times 10^{-3}$ cm. Assuming the particle to be spherical, find its volume. (S.N.F.)

$V = 4/3 \ r^3$ Use = 22/7

20. In multiplying two decimal numbers, the no. of places to the right of the decimal point in the product is the sum of the no. of places to the right of the decimal point in the two factors. Using the illustration 1.576 x 2.32, show the reason for the rule.

21. Solve for a: (a) $4^{2a} = 64$ (b) $3^a = \frac{1}{81}$ (c) $4 \times 10^c = .004 \times 10^{-6}$

22. Written down the largest number which can be written with just 3 digits and no other symbols. (The answer is not 999!)

23. At exactly 2 o'clock, two bacteria are placed in a growing medium. One minute later there are four bacteria, in another minute they have increased to 8, etc. At exactly 3 o'clock the growing mass of bacteria measures 1 gallon. At what time was there one pint of bacteria? (The answer is not 2:075)