Mathematics Support Capsules

COMPLETING THE SQUARE

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In algebra and analytic geometry it is frequently convenient to rewrite an expression with a perfect square.

Example:

$$x^2 + 6x = 0$$

adding 0 doesn't change value.

$$\underbrace{x^2 + 6x}_{(x+3)^2} + 9 - 9 = 0$$

$$(x+3)^2 - 9 = 0$$

now we can write the quadratic expression as a perfect square.

The only trick is, how do you know what to add and subtract? The answer is, of course, whatever will give you a perfect square. A sequence of steps that will do the job is a follows:

1. Collect terms in x^2 and x
2. Factor out the coefficient of x^2
3. Now, compute term to be added:

	in the above example:
take the new coefficient of x,	(6)
halve it,	$\left(\frac{6}{2} = 3\right)$
and then square the result.	$(3^2 = 9)$

4. Add this number to the quadratic expression but subtract it also so that you don't change the value of the overall expression or equation.

5. Now your quadratic can be expressed as a <u>perfect square</u>.

Another example:

$$3x^2 + 4x \qquad - y = 8$$

$$3(x^2 + \tfrac{4}{3}x \qquad) - y = 8$$

$$\underbrace{3(x^2 + \tfrac{4}{3}x}_{} \overbrace{+ \tfrac{4}{9}) - 3(\tfrac{4}{9})}^{=0} - y = 8$$

$$3(x^2 + \tfrac{2}{3})^2 \qquad - \tfrac{4}{3} - y = 8$$

or

$$3(x^2 + \tfrac{2}{3})^2 = (y + \tfrac{28}{3}) =$$

1. <u>Collect</u>
2. <u>Factor</u>
3. <u>Compute</u>
   coefficient of x: $\frac{4}{3}$
   halve it: $\frac{4}{6} = \frac{2}{3}$
   square result: $(\frac{2}{3})^2 = \frac{4}{9}$
4. <u>Add & Subtract</u>, remember the 3 you factored out. that means you're really adding $3(\frac{4}{9})$, so you must subtract the whole thing, $3(\frac{4}{9})$.
5. <u>Perfect Square</u>

This is the standard form for a parabola (See GRAPHING VIII. CONIC SECTIONS). Completing the square has put the equation in a form that is very simple to graph.

roblems on your own	$x^2+6x=5$	$x^2=4x$	$2x^2-6x=3$	$x^2-x=y$	$x^2+bx=0$
1. Collect					
2. Factor					
3. Compute					
4. Add & Subtract					
5. Perfect Square					

Uses for completing the square:
Graphing conic sections
Deriving the quadratic formula (try it, on $ax^2+bx+c=0$)
Techniques of integral calculus

Answers: A. $x^2 + 6x = 5$ add & subtract $(\frac{6}{2})^2 \Rightarrow x^2 + 6x + 9 - 9 = 5 \Rightarrow (x^2 + 6x + 9) - 9 = 5 \Rightarrow (x+3)^2 - 9 = 5$ or $(x+3)^2 = 14$.
B. $x^2 = 4x \Rightarrow (x^2 - 4x + 4) - 4 = 0 \Rightarrow (x-2)^2 - 4 = 0. \Rightarrow x^2 - 4x = 0$ add & subtract $(\frac{-4}{2})^2 \Rightarrow x^2 - 4x + 4 - 4 = 0$ C. $2x^2 - 6x = 3$
$2(x^2-3x) = 3$ add & subtract (inside the parenthesis) $(\frac{-3}{2})^2 \Rightarrow 2(x^2 - 3x + \frac{9}{4} - \frac{9}{4}) = 3 \Rightarrow 2(x^2 - 3x + \frac{9}{4}) - \frac{9}{2} = 3 \Rightarrow 2(x - \frac{3}{2})^2 - \frac{9}{2} = 3$
D. $x^2 - x = y$ add & subtract $(-\frac{1}{2})^2 \Rightarrow (x^2 - x + \frac{1}{4}) - \frac{1}{4} = y \Rightarrow (x - \frac{1}{2})^2 - \frac{1}{4} = y$. E. $x^2 + bx = 0$ add & subtract $(\frac{b}{2})^2 \Rightarrow x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} = 0 \Rightarrow (x + \frac{b}{2})^2 - \frac{b^2}{4} = 0$

You can always check answers by multiplying out your final equation and seeing if it is the same as the original.

Lastly, if you have forgotten the quadratic formula, it is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (complete the square, get in form $(x + \frac{b}{2})^2 =$. and then take $\sqrt{\ }$ both sides.)