

Row Reduction of small integer Matrices

Optimized to prevent arithmetic errors in manual calculation

Check whether the matrix is row reduced, keep integer values as small as possible.

For each row:

1. Find first non-zero element
2. If all other elements in the column = 0;
 - go to next row;
 - else make them = 0.

Permute rows to echelon form

Normalize first element in each row to 1

Start-	Simplify:	Setup:	Add:
Consider an integer matrix: $\left[\begin{array}{ccc c} 2 & 4 & 0 & 6 \\ 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & -3 \end{array} \right] /2$	Divide so that no row has a common factor. $\left[\begin{array}{ccc c} 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 5 \\ 0 & 1 & 1 & -3 \end{array} \right] \begin{array}{l} *2 \\ *-1 \\ *1 \end{array}$	Multiply to achieve the Lowest Common Multiple (L.C.M.) throughout the column: Make a positive coefficient on the element to be kept, a negative coefficient on the elements to be zeroed. $\left[\begin{array}{ccc c} +2 & 4 & 0 & 6 \\ -2 & -1 & -3 & -5 \\ 0 & 1 & 1 & -3 \end{array} \right]$	Copy the row to be kept. Add it to each of the other rows (You will never need to subtract.) $\left[\begin{array}{ccc c} 2 & 4 & 0 & 6 \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 1 & -3 \end{array} \right] \begin{array}{l} /2 \\ /1 \\ /1 \end{array}$
Work on Column 1 L.C.M. = 2	$\left[\begin{array}{ccc c} 1 & 2 & 0 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 1 & -3 \end{array} \right] \begin{array}{l} *-3 \\ *2 \\ *-6 \end{array}$		
Work on Column 2 L.C.M. = 6	$\left[\begin{array}{ccc c} 1 & 2 & 0 & 3 \\ 0 & 3 & -3 & 1 \\ 0 & 1 & 1 & -3 \end{array} \right] \begin{array}{l} *-3 \\ *2 \\ *-6 \end{array}$		
Work on Column 3 L.C.M. = 6	$\left[\begin{array}{ccc c} 3 & 0 & 6 & 7 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & 3 & -5 \end{array} \right] \begin{array}{l} *-1 \\ *2 \\ *2 \end{array}$		
End. Divide to make lead entries = 1	$\left[\begin{array}{ccc c} 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 1 & -5/3 \end{array} \right]$		
Virtues of the method: <ul style="list-style-type: none"> • Utterly deterministic – no confusion over what to do next. • Self-documenting. • Simple arithmetic: <ul style="list-style-type: none"> – Uses only +, * and integer divide– no fractions until the end. – Uses smallest possible integers often with known factors. • Easy to follow, correct, or resume if interrupted. • Minimizes errors when calculating by hand or on the board. 			

Notes for Linear Algebra:

Express as a sequence of three matrix multiplications per pivot column zeroed.

0. **Begin:** The (optionally) Augmented matrix: A_0
1. **Operation:** $Setup$ (Multiply): Add (Pivot row): $Simplify$ (Divide): $= A_{PivotCol}$
2. **Start** by *Simplifying* (Divide rows by any common factors): $D_0 A_0 = A_1$
3. **Zero Column 1:** $M_1 A_1 = P_1 M_1 A_1 = A_2$
4. **Zero Column 2:** $M_2 A_2 = P_2 M_2 A_2 = A_3$
5. **Zero Column 3:** $M_3 A_3 = P_3 M_3 A_3 = A_4$
6. **End** by *Simplifying* (Divide rows by pivot values to get 1 in pivot position): $D_4 A_4 = A_{rr}$
7. **Behold!** A_0 is now completely row-reduced to A_{rr}

Discover:

0. An invertible Diagonal Matrix can be built from the Identity Matrix using elementary row operations.
 1. The determinant of a diagonal matrix.
 2. *Simplify:* Divides rows using a diagonal matrix D_n
 3. *Setup:* Multiplies rows using diagonal matrix M_n
 4. *Add:* Adds the Pivot row n to other rows as necessary P_n
 - a. What does P_n look like?
 - b. How can P_n be built from elementary row operations?
 - c. What is the determinant of P_n ?
 5. What elementary row operation(s) is (are) not included in this development?
 - a. What is the determinant of the matrix representing such operation(s)?
 6. If A_0 Reduces to the Identity Matrix, express the inverse of A_0 as a composition of the matrices used.